NONLINEAR AND NONLOCAL DEGENERATE DIFFUSIONS ON BOUNDED DOMAINS

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We investigate quantitative properties of nonnegative solutions $u(t,x) \geq 0$ to the nonlinear fractional diffusion equation, $\partial_t u + \mathcal{L}F(u) = 0$ posed in a bounded domain, $x \in \Omega \subset \mathbb{R}^N$, with appropriate homogeneous Dirichlet boundary conditions. As $\mathcal{L}$ we can use a quite general class of linear operators that includes the three most common versions of the fractional Laplacian $(-\Delta)^s$, $0 < s < 1$, in a bounded domain with zero Dirichlet boundary conditions; many other examples are included. The nonlinearity F is assumed to be increasing and is allowed to be degenerate, the prototype being $F(u) = |u|^{m-1}u$, with $m > 1$.

First we present some result about sharp boundary behaviour and regularity for the associated stationary elliptic problem (semilinear). Next, we will shortly present some recent results about existence, uniqueness and a priori estimates for a quite large class of very weak solutions, that we call weak dual solutions. We will devote special attention to the regularity theory: decay and positivity, boundary behavior, Harnack inequalities, interior and boundary regularity, and (sharp) asymptotic behavior. All this is done in a quantitative way, based on sharp a priori estimates. Although our focus is on the fractional models, our techniques cover also the local case s = 1 and provide new results even in this setting. A surprising instance of this problem is the possible presence of nonmatching powers for the boundary behavior: for instance, when $\mathcal{L} = (-\Delta)^s$ is a spectral power of the Dirichlet Laplacian inside a smooth domain, we can prove that, whenever $2s > 1 - 1/m$, solutions behave as $dist^{1/m}$ near the boundary; on the other hand, when $2s \leq 1 - 1/m$, different solutions may exhibit different boundary behaviors even for large times. This unexpected phenomenon is a completely new feature of the nonlocal nonlinear structure of this model, and it is not present in the semilinear elliptic case. The above results are contained on a series of recent papers in collaboration with A. Figalli, Y. Sire, X. Ros-Oton and J. L. Vazquez.