COMMENTS ON SERRE, CLASSES DES CORPS CYCLOTOMIQUES

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- Theorem 2: It is now standard to call the constants μ, λ, ν.
- p. 83, last full paragraph: In modern papers, one often writes K_∞ (instead of L) for $\bigcup_{n\ge 0} K_n$.
- Section 3: Today these are more commonly called $\mathbb{Z}_p$-extensions, even though the isomorphism between the Galois group Γ and $\mathbb{Z}_p$ might not be natural.
- Lemma 4, or rather the stronger statement proved, that elements $a_1, \ldots, a_d$ generate X if and only if their images in $X/\mathfrak{m}X$ span $X/\mathfrak{m}X$ (or the special case that $X = 0$ if and only if $X = \mathfrak{m}X$), is sometimes called Nakayama's lemma for compact modules.
- Modules in $\mathcal{C}$ are called pseudo-null. Equivalent conditions for a f.g. A-module M to be pseudo-null:
  (1) $\operatorname{supp} M$ is of codimension ≥ 2 in $\operatorname{Spec} A$.
  (2) $M_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p}$ of height ≤ 1.
- The modern term for $\mathcal{C}$-isomorphism is pseudo-isomorphism, and so on.
- p. 90, last sentence: Serre is using that localization commutes with the formation of the dual, for finitely presented modules.
- p. 91, the two types of prime ideals: First, A is a UFD, so height 1 primes are principal. Since $A/(p)$ is the domain $\mathbb{F}_p[[T]]$, the element p is prime. Given any nonzero element $f = \sum a_i T^i$ of A, we may factor out the maximum power of p to assume that some a_n is a unit, in $\mathbb{Z}_p^\times$; choose n minimal such that $a_n \in \mathbb{Z}_p^\times$. Finally, use the nonarchimedean Weierstrass preparation theorem, which yields a factorization $f = Pu$, where P is a monic polynomial of degree n reducing to $T^n \bmod p$, and u is an element of $A^\times$, i.e., a power series whose constant term is in $\mathbb{Z}_p^\times$.

  Proof of the nonarchimedean Weierstrass preparation theorem: We will copy the proof of Hensel's lemma for polynomial factorizations. The factorization $f \bmod p = T^n(a_n + a_{n+1}T + \ldots)$ in $\mathbb{F}_p[[T]]$ can be lifted to a factorization mod p^2, mod p^3, etc., in which the first factor is always a monic polynomial of degree n, and the second factor is a unit. These converge to the desired factorization.

  How to remember the Weierstrass preparation theorem: The theory of Newton polygons says that the function f on the open unit disk has exactly n zeros (counting multiplicity), and P is the monic polynomial with those zeros. (Logically, this is backwards, however, since the Weierstrass preparation theorem is used to deduce the theory of Newton polygons for power series from the theory of Newton polygons for polynomials.)
- p. 92, case i: The group $X/\nu_n X = A/\nu_n A$ is infinite since $\nu_n \in pA$.

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- p. 92, case ii: The A-module generated by ν_n in $(\mathbb{Z}/p^m\mathbb{Z})[\Gamma_n]$ is a free $\mathbb{Z}/p^m\mathbb{Z}$-module generated by that element, so $c = m$.
- p. 93, first line: The last equation should be $\nu_n = \nu'_{n-1}\nu_{n-1}$.
- p. 93, last line: The submodule $Y = \nu_{n_0}X$ has the same rank as X because of the hypothesis that X/ν_nX is finite for all n.

References

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