COMMENTS ON COATES AND SUJATHA, CYCLOTOMIC FIELDS AND ZETA VALUES

BJORN POONEN

- Chapter 2: I think it is better to think of the ring $R := \mathbb{Z}_p[[T]]$ as analytic functions on the open unit disk D around 1, by setting $T = U - 1$. The mathematical content is the same, but it eliminates many of the appearances of $+1$ and -1 in the formulas. Theorem 2.1.2 then asks for $f \in R$ such that $f(\zeta_n) = u_n$ for all $n \geq 0$; here ζ_n is viewed as a U-value in D.

  If $Y \to X$ is a generically finite dominant rational map of varieties, the we obtain a finite extension of function fields $k(Y) \hookrightarrow k(X)$, and a norm operator $\mathcal{N} \colon k(Y) \to k(X)$ satisfying $(\mathcal{N}f)(x) = \prod_{y \mapsto x} f(y)$. Analogously, for the covering $D \xrightarrow{p} D$ given by the pth power map, the reverse inclusion of function rings is ϕ, sending $f(U) \in R$ to $f(U^p)$, and the $\mathcal{N}$ in Section 2.2 is the norm operator, characterized by

  $$(\mathcal{N}f)(U^p) = \prod_{\xi \in \mu_p} f(\xi U),$$

  as expected since the points ξU are the preimages of U^p under $D \xrightarrow{p} D$. Similarly, one has a trace map $\mathcal{T}$, characterized by

  $$(\mathcal{T}f)(U^p) = \sum_{\xi \in \mu_p} f(\xi U),$$

  Modulo p, the trace map for the pth power map is 0, so the image of $\mathcal{T}$ consists of power series divisible by p; then $\psi := \frac{1}{p}\mathcal{T}$.
- Equation (2.7) in terms of U is simply

  $$(\sigma f)(U) = f(U^{\chi(\sigma)}).$$

- Lemma 2.4.1 says that the functions in R of the form $f(U) - f(U^p)$ are exactly the functions in R that vanish at $U = 1$. As claimed, it is "plain" that $f(U) - f(U^p)$ vanishes at $U = 1$. For the other inclusion, given $g \in R$ vanishing at 1, the series

  $$f(U) := g(U) + g(U^p) + g(U^{p^2}) + \cdots$$

  converges in R (this requires a proof) and satisfies $f(U) - f(U^p) = g(U)$.
- Lemma 2.4.3: More precisely, this is an exact sequence of $\mathbb{Z}_p$-modules.
- Proof of Lemma 2.4.3: The argument in the first sentence actually proves $\theta^{-1}R^{\psi=0} = R^{\psi=1}$. This stronger statement is implicitly used later in the proof, when Lemma 2.4.1 is used to get exactness at $R^{\psi=0}$.

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- Proof of Lemma 2.4.3: Exactness at $R^{\psi=1}$ can also be understood as follows. The kernel of $\theta = 1 - \phi$ consists of g satisfying $g(U) = g(U^p)$. Such a g must be constant since it is constant on an infinite sequence converging to 1 (e.g., the values of U, U^p, U^{p^2}, ... for any U sufficiently close to 1).
- Definition 2.4.4: In other words,

$$\Delta = U \frac{d}{dU} \colon R^\times \to R.$$

This is the invariant derivation on the multiplicative group, just as $\frac{dU}{U}$ is the invariant differential.

References

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA

E-mail address: poonen@math.mit.edu

URL: http://math.mit.edu/~poonen/