The prismatic site

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Definition

A prism is a δ-ring A with a Cartier divisor I such that A is derived (p, I)-complete and $p \in I + \phi(I)A$. A morphism between prisms $(A, I) \to (B, J)$ is a morphism of δ-rings $A \to B$ sending I to J.

Definition

A morphism of prisms $(A, I) \to (B, J)$ is called (faithfully) flat if $B \otimes_A^L A/(p, I)$ is concentrated in degree 0 and (faithfully) flat.

Definition

A prism is called

- *perfect* if A is a perfect δ-ring, i.e. $\phi\colon A \to A$ is an isomorphism
- *bounded* if A/I has bounded p-torsion, $A/I[p^\infty] = A/I[p^c]$ for some c
- *crystalline* if $I = (p)$
- *oriented* if $I = (d)$. The choice of generator is called *orientation*

Setup

Fix a bounded prism (A, I) and a p-completely smooth A/I-algebra R, i.e. a p-complete A/I-algebra such that $R \otimes^L_{A/I} A/(p, I)$ is concentrated in degree 0 and a smooth $A/(p, I)$-algebra. For simplicity assume $I = (d)$, but let us not use that too often.

Example

The p-completion of a smooth A/I-algebra is p-completely smooth.

Goal

We want to produce a complex $\Delta_{R/A}$ with a "Frobenius" morphism $\phi_{R/A}$ such that

- $\Delta_{R/A} \otimes_A^L A/I$ is related to differential forms of R relative to A/I
- $(\Delta_{R/A}[\frac{1}{I}]^\wedge, \phi_{R/A})$ is related to the p-adic étale cohomology of $R[1/p]$

Example

Let S be a perfect $\mathbb{F}_p$-algebra, $(A, I) = (W(S), (p))$. This is a bounded prism and the prismatic theory will be equivalent to crystalline cohomology.

Example

Let $A = \mathbb{Z}_p[\![q-1]\!]$, viewed as a δ-ring via $\phi(q) = q^p$, let $[p]_q = \frac{q^p-1}{q-1} = 1 + \cdots + q^{p-1}$. Then $(A, [p]_q)$ is a bounded prism related to q-de Rham cohomology.

Definition (Prismatic site)

The *prismatic site of R relative to A*, denoted $(R/A)_{\mathbb{\Delta}}$ has

- objects: prisms (B, IB) over (A, I) with a map $R \to B/IB$ of A/I-algebras
- morphisms: the obvious ones
- topology: generated by faithfully flat maps

Write the typical object as $R \to B/IB \leftarrow B$, display it as

$$\begin{array}{ccccc} A & & \longrightarrow & & B \\ \downarrow & & & & \downarrow \\ A/I & \longrightarrow & R & \longrightarrow & B/IB \end{array}$$

Let $\mathcal{O}_{\mathbb{\Delta}}$ respectively $\overline{\mathcal{O}}_{\mathbb{\Delta}}$ the functors sending $(R \to B/IB \leftarrow B)$ to B respectively B/IB.

Remark

Any prism (B, J) over (A, I) will be of the form (B, IB)

Remark

The actual prismatic site is the opposite of what is defined above (add "Spf" everywhere). (post talk remark: It becomes confusing to switch between rings and their formal spectrum, all the statements are true as stated (I hope!) but one should exercise care when reading)

Remark

There is an obvious generalization to p-completely smooth p-adic formal schemes X. The affine case generalizes easily, also one can replace the topology with the indiscrete one (sheaves $=$ presheaves) and still compute the correct cohomology.

Remark

The last one follows from the facts:

- $\mathrm{Sh}((R/A)_{\Delta})$ is a slice topos of $\mathrm{Sh}((*/A)_{\Delta})$
- the structure sheaf on $\mathrm{Sh}((*/A)_{\Delta})$ has no higher cohomology for any prism (A, I)
- there are prismatic envelopes in the affine case, so there are weakly terminal objects (so we get morphism of topoi)

For for details in the crystalline setting see [Tag 07JK].

Remark

The prismatic site makes sense for any A/I-algebra R, but without conditions on R the resulting theory is incomputable.

Example

When $R = A/I$, the prismatic site is just all prisms over (A, I) and has an initial object given by $R = A/I \leftarrow A$.

Example

Say $R = A/I\langle X \rangle = A/I[X]_p^\wedge$. Then $(R/A)_{\Delta}$ has no initial object.
Let $A[X]$ be a δ-A-algebra via $\delta(X) = 0$. Let B be the (p, I)-adic completion of $A[X]$. Then $B/IB \cong R$. For any p-completely smooth A/I-algebra R there exists a (p, I)-completely smooth lift $\widetilde{R}$ of R to A with a δ-A-algebra structure on $\widetilde{R}$. This defines an object $R \cong \widetilde{R}/I\widetilde{R} \leftarrow \widetilde{R} \in (R/A)_{\Delta}$.

Example

Assume that (A, I) is a perfect prism. Any map $R \to S$ with S perfectoid lifts to a unique map of prisms $(A, I) \to (A_{\inf}(S), \ker(\theta_S))$. This gives a prism $(R \to S \leftarrow A_{\inf}(S)) \in (R/A)_{\Delta}$. This gives an embedding of perfectoid R-algebras into $(R/A)_{\Delta}$ with essential image those $R \to B/IB \leftarrow B$ with (B, IB) a perfect prism.

Example

Let us construct such an element. Assume that (A, I) is a perfect prism. Let $R = A/I\langle X \rangle = A/I[X]^{\wedge}_{p}$.

Take $S = A/I\langle X^{\frac{1}{p^\infty}} \rangle$. We see that $A_{\inf}(S) = A[X^{\frac{1}{p^\infty}}]^{\wedge}_{(p,I)}$, where the δ-structure is determined by requiring $\delta(X^{\frac{1}{p^n}}) = 0$. The obvious A/I-algebra $R \to S$ gives an object $R \to S \leftarrow A_{\inf}(S) \in (R/A)^{\mathrm{perf}}_{\Delta}$.

Definition

The *prismatic complex* $\Delta_{R/A}$ of R is

$$\Delta_{R/A} := R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta}) = \mathrm{RHom}_{\mathrm{Sh}((R/A)_{\Delta})}(*, \mathcal{O}_{\Delta})$$

This is a derived (p, I)-complete commutative algebra in $D(A)$; the Frobenius on $\mathcal{O}_{\Delta}$ induces a ϕ-semilinear map $\Delta_{R/A} \to \Delta_{R/A}$. The *Hodge-Tate complex* $\overline{\Delta}_{R/A}$ is

$$\overline{\Delta}_{R/A} := R\Gamma((R/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta}) = \mathrm{RHom}_{\mathrm{Sh}((R/A)_{\Delta})}(*, \overline{\mathcal{O}}_{\Delta})$$

This is a derived p-complete algebra in $D(R)$, and we have $\Delta_{R/A} \otimes^{L}_{A} A/I \cong \overline{\Delta}_{R/A}$

Example

If $R = A/I$, then $\Delta_{R/A} \cong A$ and $\overline{\Delta}_{R/A} \cong A/I$, as $(R/A)_{\Delta}$ has an initial object.

Definition

Let B be a commutative ring. A *B-dga* is given by a pair $(E^\bullet, d)$ where $E^\bullet$ is a graded B-algebra and $d\colon E^\bullet \to E^{\bullet+1}$ a B-linear map satisfying the signed Leibniz rule. It is called *graded commutative* if $ab = (-1)^{|a||b|}ba$ for homogenous elements a, b. If additionally $a^2 = 0$ for elements of odd degree it is called *strictly commutative*. This only matters in characteristic two.

Definition

Let $B \to C$ be a map of commutative rings. Denote the algebraic de Rham complex by

$$(\Omega^\bullet_{C/B}, d_{dR}) := \left(C \to \Omega^1_{C/B} \to \Omega^2_{C/B} \to \dots\right)$$

It becomes as strictly commutative B-dga.

Lemma

Let $(E^\bullet, d)$ be a graded commutative B-dga with $E^i = 0$ for $i < 0$. Assume that we are given a B-algebra map $\eta\colon C \to E^0$ such that for every $x \in C$, the element $y := d(\eta(x)) \in E^1$ satiesfies $y^2 = 0$ (This last condition is automatic if $E^\bullet$ is strictly graded commutative.) Then the map $C \to E^0$ extends uniquely to a map $\Omega^\bullet_{C/B} \to E^\bullet$.

Proof.

$E^\bullet$ is a module over E^0. The composite $C \xrightarrow{\eta} E^0 \xrightarrow{d} E^1$ is a B-linear derivation, thus extends to $\eta^1\colon \Omega^1_{C/B} \to E^1$. Extend using wedges □

Remark

We actually want to work with the continuous de Rham complex and write $\Omega^i_{R/(A/I)}$ for the derived p-completion of $\Omega^i_{R/A}$.

Remark

(Since A has bounded p^∞-torsion, derived p-completion agrees with naive derived p-completion, that is $R\lim \Omega^i_{R/A} \otimes^{\mathbb{L}}_A A/p^nA$ [Tag 0923]. As R is a formally smooth A-algebra, $\Omega^i_{R/A}$ is finitely generated projective, in particular the derived tensor product is concentrated in degree 0 and agrees with the underived tensor product. It follows that the pro-system satisfies the Mittag-Leffler condition, so the derived limit agrees with the underived one. It is a direct summand of $(A^\wedge_p)^n = A^n$, where the last equality follows because A is p-complete, as it is bounded p^∞-torsion and derived p-complete using the same argument as above. For the rank computation, develop a sheafy version of derived completions. The derived completion of a localization agrees with the localization of the derived completion, see [0A0F]). The previous lemma works when we assume that the target is derived p-complete by the universal property of derived p-completion.

Definition

For any A/I-module M and integer n define the *Breuil-Kisin twist* via $M\{n\} := M \otimes_{A/I} (I/I^2)^{\otimes n}$. If $n \geq 0$, then $M\{n\} = M \otimes_{A/I} I^n/I^{n+1}$. Observe that there is a short exact sequence

$$0 \to I^{i+1}/I^{i+2} \to I^i/I^{i+2} \to I^i/I^{i+1} \to 0$$

Tensoring with $\overline{\Delta}_{R/A} \otimes^L_{A/I} -$, we get a triangle

$$\overline{\Delta}_{R/A}\{i\} \to \overline{\Delta}_{R/A} \otimes^L_{A/I} I^i/I^{i+2} \to \overline{\Delta}_{R/A}\{i+1\}$$

in $D(R)$. Taking cohomology we get a map $\beta^i_I \colon H^i(\overline{\Delta}_{R/A})\{i\} \to H^{i+1}(\overline{\Delta}_{R/A})\{i+1\}$, which is a derivation. This makes $H^*(\overline{\Delta}_{R/A}\{*\}, \beta_I))$ a graded commutative derived p-complete A/I-dga. There is a map $\eta \colon R \to H^0(\overline{\Delta}_{R/A})$.

Remark

Assume that $I = (d)$. We have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{\Delta}}/d \xrightarrow{d\cdot} \mathcal{O}_{\mathbb{\Delta}}/d^2 \to \mathcal{O}_{\mathbb{\Delta}}/d \to 0$$

of sheaves on $(R/A)_{\mathbb{\Delta}}$.
Take cohomology to get the Bockstein map $\beta_d \colon H^i(\overline{\mathbb{\Delta}}_{R/A}) \to H^{i+1}(\overline{\mathbb{\Delta}}_{R/A})$.
This is a derivation by abstract nonsense.

Lemma

$(H^*(\overline{\Delta}_{R/A})\{*\}, \beta_I)$ is a strictly graded commutative A/I-dga. Together with the map $\eta\colon R \to H^0(\overline{\Delta}_{R/A})$, we get the unique extension

$$\eta_R^*\colon (\Omega^*_{R/(A/I)}, d_{dR}) \to (H^*(\overline{\Delta}_{R/A})\{*\}, \beta_I)$$

These are called the Hodge-Tate comparison map.

Theorem

The Hodge-Tate comparison map is an isomorphism. In particular, $\Omega^i_{R/(A/I)} \cong H^i(\overline{\Delta}_{R/A})\{i\}$ for all i, so $\overline{\Delta}_{R/A} \in D(R)$ is a perfect complex.

Lemma

Let $\mathcal{C}$ a small category admitting finite non-empty products. Let F be an abelian presheaf on $\mathcal{C}$. Assume that there is an object $X \in \mathcal{C}$ satisfying $\mathrm{Hom}(Y, X) \neq \emptyset$ for every $Y \in \mathcal{C}$ (this is called a weakly final object). Then $R\Gamma(\mathcal{C}, F)$ is computed by the complex associated to the cosimplicial abelian group

$$F(X) \to F(X \times X) \to F(X \times X \times X) \to \dots$$

obtained by applying F to the Čech nerve of X.

Proof.

The conditions say that $h_X \to *$ is a cover. See [Tag 07JM]. □

Lemma (Prismatic envelopes)

Let (B, J) be a δ-pair over (A, I). Then there is a universal map $(B, J) \to (C, K)$ of δ-pairs over (A, I) where (C, K) is a prism (and thus $K = IC$). We sometimes write $C := B\{\frac{J}{I}\}^{\wedge}$
This is to say that the inclusion $\{$prisms over $(A, I)\} \subset \delta$-pairs over (A, I) admits a left adjoint.

Proof.

Pick a generator $d \in I$. Consider the δ-ring B_0 obtained by adjoining $\frac{x}{d}$ for each $x \in J$ to B in the world of δ-A-algebras. Let B_1 be the derived (p, d)-completion (as modules) of the largest d-torsionfree quotient of B_0 in the world of δ-rings. If B_1 is d-torsionfree, then setting $C = B_1$ and $K = dB_1$ solves the problem. This is because if we map to a prism (D, dD) over (A, I), we must map J to dD, forcing us to have terms of the form $\frac{y}{d}$ for $y \in J$. If not, one transfinitely iterates the operation passing to the maximal d-torsionfree quotient in δ-rings and derived (p, d)-completion (as modules) to arrive at the required ring C □

Corollary

The category $(R/A)_{\Delta}$ admits finite non-empty coproducts.

Proof.

Let $R \to B/IB \leftarrow B$ and $R \to C/IC \leftarrow C$ be two objects in $(R/A)_{\Delta}$. Consider the δ-ring $D_0 := B \otimes_A C$. Let J be the kernel of the map

$$D_0 \to B/IB \otimes_{A/IB} C/IC \to B/IB \otimes_R C/IC$$

Then (D_0, J) is the coproduct in the category of δ-pairs (E, K) over (A, I) equipped with a map $R \to E/K$. Rewrite δ-pairs (E, K) to pairs $(E, E \twoheadrightarrow E/K)$, where the surjective map is just a morphism of rings. Then it is clear that without the condition of $R \to E/K$ that the coproduct must be $(B \otimes_A C, B/IB \otimes_{A/IA} C/IC)$. By adding the condition $R \to E/K$ we see that the coproduct must be $(B \otimes_A C, B/IB \otimes_R C/IC)$. Now to get the coproduct in $(R/A)_{\Delta}$ we must apply the prismatic envelope. $\square$

Definition (Čech-Alexander complex)

There exists a free δ-A-algebra F_0 equipped with a surjection $F_0 \to R$. Take for example the free δ-A-algebra on the underlying set $\widetilde{R}$, a (p, I)-completely smooth lift of R to A or take the δ-A-algebra $W(R)$ instead of $\widetilde{R}$ to get a strictly functorial construction. Let $J = \ker(F_0 \to R)$. Let (F, IF) over (A, I) be the prismatic envelope of (F_0, J). Moreover, there is an induced map $R \cong F_0/J \to F/IF$. Diagramatically:

$$\begin{array}{ccccc} A & \longrightarrow & F_0 & \longrightarrow & F = F\{\frac{J}{I}\}^\wedge \\ \downarrow & & \downarrow & & \downarrow \\ A/I & \longrightarrow & F_0/J \cong R & \longrightarrow & F/IF \end{array}$$

where the top arrows are δ-maps. We obtain an object $X = (R \to F/IF \leftarrow F) \in (R/A)_{\Delta}$.

Lemma

There is a map $X \to Y$ for any $Y \in (R/A)_{\Delta}$.

Proof.

By the universal property of the prismatic envelope it is sufficient to check that for any $(R \to B/IB \leftarrow B) \in (R/A)_{\Delta}$ there exists a δ-map $F_0 \to B$ compatible with the map $R \to B/IB$. As F_0 is a free δ-A-algebra, just send the generators to lifts in B of their image in $F_0 \to R \to B/IB$. Thus we can compute $\Delta_{R/A}$ by the cosimplicial δ-A-algebra

$$F^0 \to F^1 \to F^2 \to \dots$$

where $F^n = \mathcal{O}_{\Delta}(X \times \dots \times X)$ where the product is $n+1$-times. In particular, $F^0 = F$ and each F^n is d-torsion free and derived (p, d)-complete δ-ring. □

Lemma

Fix an element $t \in R$. Then $\beta_d(\eta(t)) \in H^1(\overline{\Delta}_{R/A})$ squares to 0.

Remark

Let me fix a mistake I have made during my talk:
I claimed that $H^1(\overline{\Delta}_{R/A}) = 0$ when $R = A/I\langle X\rangle = A/I[X]_p^\wedge$. This is not true! (in fact the crystalline comparison shows that it is huge). What I wanted to claim is that $H^2(\overline{\Delta}_{R/A}) = 0$ (which suffices of course). This is done how I alluded to during the talk: one shows it when $I = (p)$ using the crystalline comparison, use base change to show it for the universal oriented case, adds some work to get it for oriented prisms and then deduces the general case by using faithfully flat descent and reducing to the oriented case.

Proof.

Assume $p = 2$, the assertion is trivial otherwise. Choose a cosimplicial d-torsion free δ-A-algebra $F^\bullet$ computing $\Delta_{R/A}$. Write $\partial(-)$ for the alternating sum of the face maps, so this is the differential in the associated cochain complex. Note that $\partial(-)$ is A-linear and commutes with ϕ, as this holds for each of the face maps. Choose some $T \in F^0$ lifting $\eta(t) \in F/IF$. Write $U, V \in F^1$ and $X, Y, Z \in F^2$ for the images of T under the various structure maps $F^0 \to F^1$ and $F^0 \to F^2$ in $F^\bullet$. So $\partial(T) = U - V$ and $\partial(U) = X - Y + Z$. There is a unique $\alpha \in F^1$ such that $U - V = da$ (this d is the generator of I, not a differential). Since $U - V$ is a cycle (even a boundary) and F^2 is d-torsion free, α is also a cycle. The image of α in $H^1(F^\bullet/dF^\bullet)$ is $\beta_d(\eta(t))$ by definition. Thus we need to check $\alpha \cup \alpha = 0$ in $H^2(F^\bullet/dF^\bullet)$. Equivalently it suffices to see that $(U - V) \cup (U - V)$ in $H^2(d^2F^\bullet/d^3F^\bullet)$ vanishes. □

Proof.

By definition this is computed by $(X-Y)(Y-Z) \in d^2F^2$. It will suffice to show that $(X-Y)(Y-Z) = \partial(d^2\delta(\alpha))$. Since $p=2$, check that

$$\delta(a-b) = \delta(a) - \delta(b) + b(a-b)$$

holds. Taking $a=U$ and $b=V$ and noting that $\delta(U) - \delta(V) = \partial(\delta(T))$, we get

$$\delta(U-V) = \epsilon + V(U-V)$$

where $\epsilon \in F^1$ is a boundary. Applying the differential gives

$$\begin{aligned}\partial(\delta(U-V)) &= \partial(V(U-V)) \\ &= Y(X-Y) + Z(Y-Z) - Z(X-Z) \\ &= (X-Y)(Y-Z) \in d^2F^2\end{aligned}$$

Now we need to see that the left hand side with $\partial(d^2\delta(\alpha))$ (no reason to believe that $\delta(U-V) \in d^2F^2$). □

Proof.

Apply $\delta(-)$ to $U - V = d\alpha$ to get

$$\delta(U - V) = d^2\delta(\alpha) + \phi(\alpha)\delta(d)$$

So it suffices to see that $\phi(\alpha)$ is a cycle, but α is a cycle and ϕ commutes with the differential. □