Perfectoidization and perfect prismatic complex $\Delta_{X/A,\mathrm{perf}}$

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Outline

1. What is perfection in char p..?
2. Construction via prismatic cohomology
3. Universal property, applications

Main players

S p-complete ring, $X = \mathrm{Sp}(S)$.

Choose a perfect prism (A, I), with $A/I \to S$

$\rightsquigarrow \Delta_{X/A} \in D(A)$, $\Delta_{X/A,\mathrm{perf}} := (\mathrm{colim}_\phi \Delta_{X/A})^\wedge \in D_{(p,I)-comp}(A)$.

The (derived) "perfectoidization" of S

$S_{\mathrm{perfd}} := \Delta_{X/A,\mathrm{perf}} \otimes^L_A A/I \in D_{p-\mathrm{comp}}(S)$.

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Why the name? Is S_{perfd} (derived) perfectoid / independent of (A, I) / the classical perfection in char p case? Is S_{perfd} **universal** ?

How do we study S_{perfd}?

- HT comparison + goodness of $L_{X/(A/I)}$ + derived Nakayama $\rightsquigarrow$ control $\Delta_{X/A}$ hence S_{perfd}, get descent and base change.
- Study general S via reduction to nice $A/I \to S$. For nice S (e.g quasiregular semi-perfectoid, in particular perfectoid), show S_{perfd} is a classical universal perfectoid over S.

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- Study general S via reduction to nice $A/I \to S$. For nice S (e.g quasiregular semi-perfectoid, in particular perfectoid), show S_{perfd} is a classical universal perfectoid over S.
- Universality of S_{perfd} comes from **derived "functorial" universality** of $\Delta_{X/A}$. $\Delta_{X/A}$ has a derived δ-ring (even derived prism) structure (to see this, we must use simplicial and derived tools, not just classical rings).

So $\Delta_{X/A,\text{perf}}$ is a derived perfect δ-ring $\rightsquigarrow$ $\Delta_{X/A,\text{perf}}$ mod I is derived perfectoid.

3 levels of universality in category theory

$X \in \mathcal{C}$, weakly initial $<$ "functorial" initial $<$ initial (uniqueness).

"Functorial": for any $Y \in \mathcal{C}$, There is a map $X \to Y$, functorial on Y.

Initial object $\times$ something can be "functorial" initial.

- By design, $R\Gamma(-, \mathcal{O}) = R\lim(*)$ of a site, hence $\Delta_{X/A}$, is **only good for** "functorial" universality (**provided** $R\Gamma(-, \mathcal{O})$ **is still in the site**).
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- Upshot: Via computations on simple examples, still get universality of $S \to S_{\mathrm{perfd}}$, and prove things needed for comparisons. The proof is delicate.

Applications

Why is perfectoidization powerful? Many applications via descent:

- Zariski closed=Strongly Zariski closed: If S is semiperfectoid, then $S \to S_{\text{perfd}}$ is surjective and universal among $S \to$ perfds .
- (Universal property of S_{perfd} for general S) $\Delta_{X/A,\text{perf}} \cong R\Gamma((X/A)^{\text{perf}}_{\Delta}, O_{\Delta})$.
- Arc descent for $S \to S_{\text{perfd}}$ (next time).
- étale comparison for perfect prisms $R\Gamma_{et}(X_\eta, \mathbb{Z}/p^n) \cong (\Delta_{X/A}[1/d]/p^n)^{\phi=1}$, can be proved via descent to the perfectoidization (next time).

References

- (Main, Lemma 2.34, 3.7 (2), 3.9, 3.13, 4.8, 4.14, 4.17-4.18, Section 6, 7, 8 of) B. Bhatt, P. Scholze, Prisms and Prismatic Cohomology, arxiv:1905.08229v3 (version 3, April 2021).
- (Main, Lecture VI, VII, VIII, IX of) B. Bhatt, Lectures on Prismatic cohomology, Fall 2018.
- (Section 3 of) B. Bhatt, Cohen-Macaulayness of absolute integral closures, Arxiv:2008.08070v1.
- (Section 2 of) D. Kubrak, A. Prikhodko, p-adic Hodge theory for Artin stacks, Arxiv:2105.05319.

Perfection in char p

k perfect field of char $p > 0$. The forgetful functor

$\{\text{perfect ring over } k\} \to \{\text{rings over } k\}$ has a left adjoint:

The perfection of a k-algebra R

$R_{\text{perf}} := \text{colim}(R \stackrel{\phi}{\to} R \stackrel{\phi}{\to} R \stackrel{\phi}{\to} ..)$, where $\phi : x \to x^p$.

Perfect algebraic Geometry

Observations (**to be generalized later**):

- R_{perf} is a R-algebra via first term of colim, or by adjointness.
- As an algebra, R_{perf} is independent of k.
- $R \to R_{\text{perf}}$ is the universal map from R to a perfect k-algebra.
- Zariski closed=Strongly Zariski closed: $R_1 \twoheadrightarrow R$ surjective with R_1 perfect, then $R \to R_{\text{perf}}$ is surjective.
- perfect rings are reduced. Frobenius is zero on higher $\pi_i, i > 0$.

Reconstruction via derived de Rham cohomology

$\mathrm{dR}_{-/k}$ = left Kan extension of de Rham complex $\Omega^*_{-/k}$ on polynomial k-algebras. ϕ on $R \rightsquigarrow \phi_k$-semilinear endomorphism $\phi_R : \mathrm{dR}_{R/k} \to \mathrm{dR}_{R/k}$.

The perfection of $\mathrm{dR}_{R/k}$

$\mathrm{dR}_{R/k,perf} := \mathrm{colim}(\mathrm{dR}_{R/k} \overset{\phi_R}{\to} \mathrm{dR}_{R/k} \overset{\phi_R}{\to} \mathrm{dR}_{R/k} \overset{\phi_R}{\to} ..)$.

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The projection $\mathrm{dR}_{R/k} \to R$ gives $\mathrm{dR}_{R/k,perf} \cong R_{perf}$: we reduce to the case R is a polynomial algebra, then $d(x^p) = px^{p-1}dx = 0$, so colimits of $\Omega^i_{R/k}$ $(i > 0)$ under ϕ_R is zero.

Reconstruction via derived prismatic cohomology

$(A, I) = (W(k), p)$ the perfect prism corresponding to k. R a $A/I = k$ algebra. $\Delta_{R/A} \in D(A)$, with $R \to \overline{\Delta}_{R/A}$. $I = (p)$, $\phi(p) \subseteq (p)$, ϕ still acts on $\overline{\Delta}_{R/A}$.

Proposition

The map $R \to \overline{\Delta}_{R/A}$ gives $\overline{\Delta}_{R/A,\mathrm{perf}} \cong R_{\mathrm{perf}}$.

Reconstruction via derived prismatic cohomology

$(A,I)=(W(k),p)$ the perfect prism corresponding to k. R a $A/I=k$ algebra. $\Delta_{R/A}\in D(A)$, with $R\to\overline{\Delta}_{R/A}$. $I=(p)$, $\phi(p)\subseteq(p)$, ϕ still acts on $\overline{\Delta}_{R/A}$.

Proposition

The map $R\to\overline{\Delta}_{R/A}$ gives $\overline{\Delta}_{R/A,\mathrm{perf}}\cong R_{\mathrm{perf}}$.

WLOG R is a polynomial algebra. By Hodge-Tate comparison $gr_i^{HT}(\overline{\Delta}_{R/A})=\Omega^i_{R/k}$, only need to check $gr_i^{HT}(\phi)=0, i>0$, $gr_0^{HT}(\phi)=\phi_R, i>0$. WLOG $R=k[x]$. By crystalline comparison, reduce to de Rham cohomology of $\mathbb{A}^1_W$ over W, mod p Hodge-Tate filtraion=canonical filtration. Then it's standard.

Prismatic cohomology

Conceptual leap: this doesn't use the Frobenius on R, only Frobenius on the test objects.

S p-complete ring, $X = \mathrm{Sp}(S)$. Choose a perfect prism (A, I), with $A/I \to S$.

The prismatic site

$(X/A)_{\Delta}$ is the opposite of the category of prisms (B, J) with a map $(A, I) \to (B, J)$ and a map $\mathrm{Sp}(B/J) \to X$ over $\mathrm{Sp}(A/I)$.

$\mathcal{O}_{\Delta} : (B, J) \to B$. $\overline{\mathcal{O}_{\Delta}} : (B, J) \to B/J = B/IB$.

$\rightsquigarrow \Delta_{X/A} = R\Gamma((X/A)_{\Delta}, \mathcal{O}_{\Delta}) \in D(A)$, with $S \to \overline{\Delta}_{X/A}$.

The powerful Hodge-Tate filtration

Proposition

$\overline{\Delta}_{X/A} := \Delta_{X/A} \otimes^L_A A/I$ admits an increasing $\mathbb{N}$-indexed filtration with gr^i given by sending $R$to the derived p-completion of $\wedge^i L_{R/(A/I)}{-i}[-i]$.

Idea: Universal property of the de Rham complex $\rightsquigarrow$ the map from de Rham to Hodge-Tate in Hodge-Tate comparison.

Remind of Hodge-Tate in char p

k perfect field char $p > 0$. X over k smooth, relative Frobenius $F : X \to X^{(p)}$. Two filtration on de Rham complex $\Omega^*_{X/k}$:

- Hodge filtration=stupid filtration $\rightsquigarrow$ Hodge spectral seq $E_1^{pq} = H^q(X, \Omega^p_{X/k})$.
- Conjugate filtraion=canonical filtration $\rightsquigarrow$ Hodge-Tate spectral seq $E_2^{pq} = H^p(X, H^q(\Omega^*_{X/k}))$.

Hodge-Tate filtration is a generalization of conjugate filtration, via Cartier isomorphism.

Perfection in mixed characteristic

$\Delta_{X/A,\mathrm{perf}} := (\mathrm{colim}_\phi \Delta_{X/A})^\wedge \in D_{(p,I)-comp}(A)$.

The (derived) "perfectoidization" of S

$S_{\mathrm{perfd}} := \Delta_{X/A,\mathrm{perf}} \otimes^L_A A/I \in D_{p-\mathrm{comp}}(S)$ a commutative algebra object.

It can be non-discrete: $S = A/I[x^{\pm 1}]^\wedge$ for the perfection (A, I) of the q-prism.

The universal classical perfectoid of S

$S_{\mathrm{perfd}'}$:= the universal classical perfectoid over S (**if it exists**), i.e $S_{\mathrm{perfd}'}$ is perfectoid, and for any $S \to R$ with R perfectoid, there is a **unique** map $S_{\mathrm{perfd}'} \to R$ extending it.

Independence of base

Proposition

Let $(A, I) \to (B, J)$ be a map of perfect prisms, and let S be a p-complete B/J-algebra. Then the natural map gives an isomorphism $\Delta_{S/A} \cong \Delta_{S/B}$. In particular, $\Delta_{S/A,\text{perf}} \cong \Delta_{S/B,\text{perf}}$, S_{perfd} is independent of (A, I).

Proof.

Use HT comparison. $L^{\wedge}_{(B/J)/(A/I)} = 0$ vanishes as A/I and B/J are both perfectoid. □

Base change

The formation of $\Delta_{X/A}$ commutes with base change in the sense that for any map of bounded prisms $(A, I) \to (B, J)$, $\Delta_{X_B/B} = B \otimes_A^L \Delta_{X/A}$. We can check directly, $S \to S_{\mathrm{perfd}'}$ also commutes with base change of the perfect prism.

$S_{\mathrm{perfd}} = S = S_{\mathrm{perfd}'}$ if S is perfectoid

So if S perfectoid, $(S/A)_\Delta$ has an object $(A_{inf}(S), \mathrm{Ker}\theta_S)$, hence a map $\Delta_{S/A} \to A_{inf}(S)$, it's an isomorphism: apply derived Nakayama and HT, done by $L^\wedge_{S/(A/I)} = 0$. We see $S_{\mathrm{perfd}} = S$, in particular discrete. Note $(A_{inf}(S), \mathrm{Ker}\theta_S)$ is also the initial object in $(S/A)_\Delta$:

Proposition

Let (A, I) be a perfect prism corresponding to a perfectoid ring $R = A/I$. Then for any prism (B, J), any map $A/I \to B/J$ of commutative rings lifts uniquely to a map $(A, I) \to (B, J)$ of prisms.

Proof: the relation between deformation theory and cotangent complex, done by $L^\wedge_{A/\mathbb{Z}_p} = 0$.

The case S is semiperfectoid

Proposition

If S is semiperfectoid i.e there is a surjection $R \to S$ with R perfectoid, then $S_{\mathrm{perfd}'}$ exists.

Proof.

Cut out the perfect prism for $S_{\mathrm{perfd}'}$ using the perfect prism $(A_{inf}(R), d)$ for R ($R_{\mathrm{perfd}'} = R = R_{\mathrm{perfd}}$), and the kernel of $A_{inf}(R) \to S$. We need to do transfinite induction, to make it d-torsion free and derived complete. □

Derived "functorial" universality of $\Delta_{X/A}$

Proposition

Assume $\overline{\Delta}_{X/A}$ is concentrated in degree zero. Then the pair $(\Delta_{X/A}, I\Delta_{X/A})$ gives a prism over (A, I), with a map $R \to \overline{\Delta_{X/A}}$. For any prism (B, J) over (A, I) equipped with a map $R \to B/J$, there is a map $\Delta_{X/A}, I\Delta_{X/A} \to (B, J)$, functorial on (B, J).

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Proof.

$\Delta_{X/A} = R\Gamma((X/A)_\Delta, \mathcal{O}_\Delta) = R\lim_{(B,J)}(B, J)$. Use Cech-Alexander complexes. and the canonical simplicial resolution of X, one gets the derived δstructure i.e a section of $W_2(-) \to (-)$ on $\Delta_{X/A}$. Here by assumption $\Delta_{X/A}$ is discrete, so $(\Delta_{X/A}, I\Delta_{X/A})$ gives a prism over (A, I). The universality is clear by definition of $R\Gamma$. □

Discreteness of perfection when S is semiperfectoid

For any X, $\Delta_{X/A,perf}$ always lies in $D^{\geq 0}$, i.e. we have $H^i = 0, i < 0$.

Frobenius is zero on higher $\pi_i, i > 0$.

If S is semiperfectoid, then $\Omega^1_{S/(A/I)} = 0$ hence $L_{X/A}[-1], \Delta_{X/A,\mathrm{perf}} \in D^{\leq 0}$ by HT comparison. So $\Delta_{X/A,\mathrm{perf}}$ is discrete, and a classical perfect δ-ring by previous proposition. It's d-torsion free, where d is the distinguished element. Hence S_{perfd} is discrete and perfectoid. By equivalence of perfectoid rings and perfect prisms, we see

Proposition

If S is semiperfectoid, then $S \to S_{\mathrm{perfd}}$ satisfies "functorial" universality among $S \to perfds$, in particular a natural map $S_{\mathrm{perfd}} \to S_{\mathrm{perfd}'}$.

Andre's Flatness lemma

Proposition

Let R be a perfectoid ring. For any set $\{f_s \in R\}_{s \in I}$ of elements of R, there exists a p-completely faithfully flat map $R \to R_\infty$ of perfectoid rings such that each f_s admits a compatible system of p-power roots in R_∞. In other words, the map $\# : R^\flat \to R$ is surjective locally for the the p-completely flat topology.

Andre's Flatness lemma

WLOG $\#I = 1$. Let S be the p-adic completion of $R[x^{1/p^\infty}]/(x - f)$, so $R \to S$ is p-completely faithfully flat. We reduce the problem to S, but S is not perfectoid in general, just semiperfectoid.

Let (A, I) be the perfect prism corresponding to R. We shall show that S_{perfd} solves the problem. We know discreteness and perfectoidness, now only need to check S_{perfd} is p-completely faithfully flat over R.

Andre's Flatness lemma

Only need to check $A \to \Delta_{S/A,\mathrm{perf}}$ is (p, I)-completely faithfully flat.

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Use HT filtration, only need to check $L_{S/R}[-1]$ (noting $\wedge^i L_{S/R}[-i] = \wedge^i(L_{S/R}[-1])$).

$R \to R[x^{1/p^\infty}] = R' \twoheadrightarrow S$. $L^\wedge_{R'/R} = 0$ by perfectoidness. We only need to show $\wedge^i L_{S/R'}[-i]$ is p-completely faithfully flat over R. But S is the quotient of R' by non-zero divisor $x - f$, so $L_{S/R'}[-1]$ is simply isomorphic to S, hence p-completely flat over R. We're done.

Zariski closed=Strongly Zariski closed

Proposition

Let R be a perfectoid ring, and let $S = R/J$ be a p-complete quotient (so S is semiperfectoid). Then there is a universal map $S \to S'$ with S' being a perfectoid ring. Moreover, this map is surjective.

We just check $S \to S_{\mathrm{perfd}}$ is surjective.

Zariski closed=Strongly Zariski closed

Assume first that the kernel $J \subseteq R$ of $R \to S$ is the p-completion of an ideal generated by a set $\{x_i\}$ of elements that lie in the image of the map $\# : R^\flat \to R$.

In this case, if J_∞ denotes the p-completion of the ideal generated by $x^{1/p^n\#}$, then check directly that the R/J_∞ is perfectoid, and $S \cong R/J \to R/J_\infty$ is the universal map from S to a perfectoid ring.

Zariski closed=Strongly Zariski closed

This proves the assertion in this case. In general, as the surjectivity of a map of p-complete R-modules can be detected after p-completely faithfully flat base change (and S_{perfd} commutes with base change), we reduce to previous case by Andre's flatness lemma.

Completely flatness

A complex M of A-modules is I-completely flat if for any I-torsion A-module N, the derived tensor product $M \otimes^L AN$ is concentrated in degree 0. This implies in particular that $M \otimes^L_A A/I$ is concentrated in degree 0, and is a flat A/I-module.

Proposition

Let (A, I) be a bounded prism. For any (p, I)-completely flat A-complex $M \in D(A)$. Then M is discrete and classically (p, I)-complete.

True universality of $S \to S_{\text{perfd}}$

Thank you!