A "10-line" proof of the Weil conjecture

Zhiyu Zhang

STAGE seminar

November 16th, 2020

Outline

1. Weil conjecture
2. Rapoport-Zink weight spectral sequence
3. The end of the proof

Motivation

A basic question in number theory is, if we have a number of polynomials $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$, how to find or just count solutions of $\{f_i(x) = 0\}$ over rings like $\mathbb{Z}$ or $\mathbb{Q}$?
One may try mod p ($\mathbb{Z} \to \mathbb{Z}/p$) or p-adic method ($\mathbb{Q} \to \mathbb{Q}_p$). It's interesting to count solutions over finite fields beyond $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and the answer will have some beautiful and uniform patterns, revealing the topology of the space.
Let's look at some examples:

Motivation

A basic question in number theory is, if we have a number of polynomials $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$, how to find or just count solutions of $\{f_i(x) = 0\}$ over rings like $\mathbb{Z}$ or $\mathbb{Q}$?
One may try mod p ($\mathbb{Z} \to \mathbb{Z}/p$) or p-adic method ($\mathbb{Q} \to \mathbb{Q}_p$). It's interesting to count solutions over finite fields beyond $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and the answer will have some beautiful and uniform patterns, revealing the topology of the space.
Let's look at some examples:

$$\#\mathbb{P}^n(\mathbb{F}_{q^m}) = 1 + q^m + q^{2m} + \ldots + q^{mn}$$

$$\#GL_n(\mathbb{F}_{q^m}) = (q^{mn} - 1)(q^{mn} - q)...(q^{mn} - q^{m(n-1)})$$

$$\#E(\mathbb{F}_{q^m}) = 1 + q^m - \alpha^m - \beta^m \text{ (E elliptic curve, } \alpha\beta = q, |\alpha| = |\beta| = q^{1/2})$$

$$\#\{A \in M_n(\mathbb{F}_{q^m}) | A^n = 0\} = q^{m(n^2-n)}$$

Weil also computed the number of $\mathbb{F}_{q^m}$-points for Fermat hypersurfaces using Jacobi sums, which is also of the form $\sum a_i^m - b_j^m$ for some specific algebraic integers a_i, b_j. This motivates the Weil conjecture:

Weil conjecture

Weil conjecture

Let X_0 be a n-dimensional smooth projective variety over a finite field $k = \mathbb{F}_q$. Then there exists algebraic integers $\alpha_{i,j} \in \overline{\mathbb{Z}}, (0 \leq i \leq 2n, 0 \leq j \leq h^i - 1)$ such that

1. (Trace formula) $\#X_0(\mathbb{F}_{q^m}) = \sum(-1)^i \alpha_{ij}^m$, for all m.
2. (Poincare duality) For fixed i, $\{q^{2n}/\alpha_{i,j}\}$ is the same as $\{\alpha_{2n-i,j}\}$ as multisets.
3. (Purity) $\forall \tau : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $|\tau(\alpha_{ij})| = q^{\frac{i}{2}}$.

Weil conjecture

Let X_0 be a n-dimensional smooth projective variety over a finite field $k = \mathbb{F}_q$. Then there exists algebraic integers $\alpha_{i,j} \in \overline{\mathbb{Z}}, (0 \leq i \leq 2n, 0 \leq j \leq h^i - 1)$ such that

1. (Trace formula) $\#X_0(\mathbb{F}_{q^m}) = \sum(-1)^i \alpha_{ij}^m$, for all m.
2. (Poincare duality) For fixed i, $\{q^{2n}/\alpha_{i,j}\}$ is the same as $\{\alpha_{2n-i,j}\}$ as multisets.
3. (Purity) $\forall \tau : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $|\tau(\alpha_{ij})| = q^{\frac{i}{2}}$.

A naive idea: $X_0(\mathbb{F}_{q^m})$ is the fixed point set of Frobenius map Fr_q^m on $X(\bar{k})$. If we have a cohomology theory for $X_{\bar{k}}$ that looks like singular cohomology for good topological spaces (locally contractible), then Lefschetz trace formula and Poincare duality will give the first and second parts. For any prime number $\ell \neq p$, Grothendieck developed a new topology i.e étale topology (so any variety is "locally contractible" (not really..) by some vanishing theorems of Galois cohomology), and the *ℓ-adic* étale cohomology theory $H^*(X_{\bar{k}}, \mathbb{Q}_\ell)$.

The Riemannian hypothesis for X_0

Analogous to the Riemannian zeta function, one can formulate the zeta function of X_0 (or any piece "$H^i(X)$"):

$$\zeta(X,s) = \exp\left(\sum_{m=1}^{\infty} \frac{\#X(\mathbb{F}_{q^m})}{m} q^{-ms}\right)$$

It's a rational function by Lefschetz trace formula. The purity is equivalent to some sort of Riemannian hypothesis. Now the real question is:
How to show the purity of the Frobenius action on $H^*(X)$, if you can't compute the action on the cohomology explicitly?

The Riemannian hypothesis for X_0

Analogous to the Riemannian zeta function, one can formulate the zeta function of X_0 (or any piece "$H^i(X)$"):

$$\zeta(X,s) = \exp\left(\sum_{m=1}^{\infty} \frac{\#X(\mathbb{F}_{q^m})}{m} q^{-ms}\right)$$

It's a rational function by Lefschetz trace formula. The purity is equivalent to some sort of Riemannian hypothesis. Now the real question is:
How to show the purity of the Frobenius action on $H^*(X)$, if you can't compute the action on the cohomology explicitly?
Weil proved it for curve C using Hodge index theorem on the surface $C \times C$ ($\#X_0(\mathbb{F}_{q^m})$ is exactly the intersection number of the graph of $Frob_q^m$ with the diagonal inside $X \times X$).
For higher dimension X, the Hodge index type results are unknown and is one of the standard conjectures. Grothendieck has a proof of purity in general using his standard conjectures.

Too naive approaches

March 31, 1964 JEAN-PIERRE SERRE

Dear Grothendieck,

I have constructed J.-P. Serre : Grothendieck had hoped to prove the Weil conjectures by showing that every variety is birationally a quotient of a product of curves.

In the present letter, I construct a counterexample in dimension 2. There are certainly simpler ones! an example of a surface whose function field is not contained in that of a product of two curves (nor, of course, of a product of n curves, since the case of n can be trivially reduced to that of 2).

I start with an abelian variety with origin 0 and an irreducible subvariety S of A of dimension 2, passing through 0, non-singular at this point and having the following bizarre property:

(*) If C and C' are two irreducible curves passing through 0 contained in S, then the sum $C + C'$ (given by the composition law on A) is not contained in S.

Too naive approaches

March 31, 1964 JEAN-PIERRE SERRE

Dear Grothendieck,

I have constructed J.-P. Serre : Grothendieck had hoped to prove the Weil conjectures by showing that every variety is birationally a quotient of a product of curves.

In the present letter, I construct a counterexample in dimension 2. There are certainly simpler ones! an example of a surface whose function field is not contained in that of a product of two curves (nor, of course, of a product of n curves, since the case of n can be trivially reduced to that of 2).

I start with an abelian variety with origin 0 and an irreducible subvariety S of A of dimension 2, passing through 0, non-singular at this point and having the following bizarre property:

(*) If C and C' are two irreducible curves passing through 0 contained in S, then the sum $C + C'$ (given by the composition law on A) is not contained in S.

Another naive approach (**Today**): every variety is indeed birational to a hypersurface in projective space. But this hypersurface can be highly singular.

Deligne's approach and Rankin method

Later, Degline proves the purity by developing many new tools: a general theory of weights, use of L-functions and Rankin–Selberg method to bound the weights.. The Lefschetz pencil is a geometric tool for induction, and its vanishing cycles are computed in SGA7. The machinery in Weil II is heavy but very powerful.

Deligne's approach and Rankin method

Later, Degline proves the purity by developing many new tools: a general theory of weights, use of L-functions and Rankin–Selberg method to bound the weights.. The Lefschetz pencil is a geometric tool for induction, and its vanishing cycles are computed in SGA7. The machinery in Weil II is heavy but very powerful.
Informally, Rankin method is the idea that you can bound the coefficient of a modular form more efficiently if you know the symmetric square L function is again automorphic (and analytic on $\Re s > 1$). Such ideas have many applications to the Ramanujan conjecture. Also, one powerful application of Weil II is the Ramanujan conjecture for $\Delta(z) = q \prod_{n>0} (1 - q^n)^{24}$.
Laumon, Brylinski and Katz simplify the proof of Deligne by using ℓ-adic Fourier transform on affine line and Plancherel identity.
Today, we will give a simple proof of the Weil conjecture (but not Weil II), based on works of Katz, Scholl, Deligne... No Lefschetz pencil, no Fourier transform is used, but we still need L-functions and Rakin method.

Strategy Today

The "10-line" proof of Weil conjecture is as follows:

- Using things like $|\sum_{n=1}^{p} \chi(a) e^{\frac{2\pi i n}{p}}| = p^{1/2}$, we can prove purity for a specific smooth hypersurface with given degree d and dimension n.
- For any local system $\mathcal{F}$ on U_0, Rankin method implies the persistence of purity in a smooth proper family.
- By deformation, we get purity for all smooth hypersurfaces.

Strategy Today

The "10-line" proof of Weil conjecture is as follows:

- Using things like $|\sum_{n=1}^{p}\chi(a)e^{\frac{2\pi i n}{p}}| = p^{1/2}$, we can prove purity for a specific smooth hypersurface with given degree d and dimension n.
- For any local system $\mathcal{F}$ on U_0, Rankin method implies the persistence of purity in a smooth proper family.
- By deformation, we get purity for all smooth hypersurfaces.
- Any variety is birational to a hypersurface X_s, deform X_s into smooth hypersurfaces, get a family over a smooth curve with one singular fiber.
- By alteration, we can assume the family is of strictly semi-stable situation.
- (Weight-monodromy) monodromy filtration for $H^*(X_{\bar{\eta}})$ has pure pieces.
- (**Today**) Weight spectral sequence relates these pieces with cohomology of (intersections of irreducible components of) $X_{\bar{s}}$, we're done by induction.

Vanishing cycles and nearby cycles

Let R be a complete DVR with finite residue field k, and $X \to R$ proper with $i : X_s \hookrightarrow X, j : X_\eta \hookrightarrow X$. If X/R is smooth, then by smooth and proper base change, $H^i(X_{\bar{s}}) \cong H^i(X_{\bar{\eta}})$ as Galois representation, in particular **the inertia group acts trivially**.

In general, how do you relate the cohomology of the special fiber and generic fiber?

The idea is to think $X \to R$ as a degeneration from X_η to X_s. We need to work with geometric fibers so will do a base change to R^{ur}. Consider $H_*(X_{\bar{\eta}}) \to H_*(X_{\bar{R}}) \cong H_*(X_{\bar{s}})$ (proper base change), by dual we get the **specialization map** $H^i(X_{\bar{s}}) \to H^i(X_{\bar{\eta}})$.

Vanishing cycles and nearby cycles

Let R be a complete DVR with finite residue field k, and $X \to R$ proper with $i : X_s \hookrightarrow X, j : X_\eta \hookrightarrow X$. If X/R is smooth, then by smooth and proper base change, $H^i(X_{\bar{s}}) \cong H^i(X_{\bar{\eta}})$ as Galois representation, in particular **the inertia group acts trivially**.

In general, how do you relate the cohomology of the special fiber and generic fiber?

The idea is to think $X \to R$ as a degeneration from X_η to X_s. We need to work with geometric fibers so will do a base change to R^{ur}. Consider $H_*(X_{\bar{\eta}}) \to H_*(X_{\bar{R}}) \cong H_*(X_{\bar{s}})$ (proper base change), by dual we get the **specialization map** $H^i(X_{\bar{s}}) \to H^i(X_{\bar{\eta}})$.

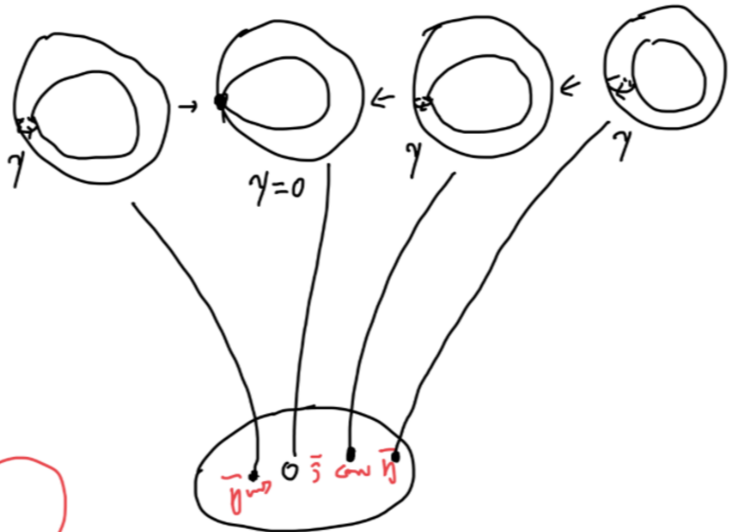
It's not an isomoprhism in general, because in such degeneration progress, some cycles will "shrink" into low dimension cycles hence vanish in the homology, they're called "vanishing cycles" $R\Phi\Lambda \in D(X_{\bar{s}})$. What will remain is called "nearby cycles" $R\Psi := \bar{i}^* R\bar{j}_*\Lambda \in D(X_{\bar{s}})$ (**definition**). By design,

Proposition

$H^i(X_{\bar{\eta}}, \Lambda) \cong H^i(X_{\bar{s}}, R\Psi\Lambda)$, *and there is a long exact sequence*

$$... \to H^{i-1}(X_{\bar{s}}, R\Phi\Lambda) \to H^i(X_{\bar{s}}, \Lambda) \to H^i(X_{\bar{s}}, R\Psi\Lambda) \to H^i(X_{\bar{s}}, R\Phi\Lambda) \to ...$$

A picture



$T \in \pi_1(R, \bar{\eta})$ "≈" $\mathbb{Z}$

$$y^2 = x(x-1)(x-\lambda)$$

$$\lambda \to 0$$

T ↻ $H^1(X_{\bar{\eta}}) \simeq \mathbb{Q}_l^2$

$H^1(X_{\bar{s}}) \simeq \mathbb{Q}_l$

$N = \log(T-1)$

monodromy operator

Weight spectral sequence

If one can construct a resolution or just a filtration of the nearby cycle in the derived category, then we may relate $R\Psi\Lambda$ with several $i_*\Lambda$ where $i : Z \hookrightarrow X_s$ is a closed immersion, hence relate the cohomology of the special fiber and generic fiber.
Let X be a strictly semi-stable scheme over R, and assume X is projective of relative dimension n. Here "strictly semi-stable" means that X is Zariski locally étale over $\operatorname{Spec} R[t_1, \dots, t_n]/(t_1..t_k - \pi)$.
Then $X_k = \bigcup_{i=1}^m X_i$, the irreducible components X_i are smooth and projective. **We take the disjoint union** $X^{(p)} := \coprod_{I \subseteq \{1,,m\},|I|=p+1} X_I$, where $X_I = \cap_{i \in I} X_i$ is smooth projective of dimension $n - p$.

Weight spectral sequence

If one can construct a resolution or just a filtration of the nearby cycle in the derived category, then we may relate $R\Psi\Lambda$ with several $i_*\Lambda$ where $i : Z \hookrightarrow X_s$ is a closed immersion, hence relate the cohomology of the special fiber and generic fiber.
Let X be a strictly semi-stable scheme over R, and assume X is projective of relative dimension n. Here "strictly semi-stable" means that X is Zariski locally étale over $\operatorname{Spec} R[t_1, \dots, t_n]/(t_1..t_k - \pi)$.
Then $X_k = \bigcup_{i=1}^m X_i$, the irreducible components X_i are smooth and projective. **We take the disjoint union** $X^{(p)} := \coprod_{I\subseteq\{1,,m\},|I|=p+1} X_I$, where $X_I = \cap_{i\in I} X_i$ is smooth projective of dimension $n - p$.

Theorem (Rapoport–Zink, Saito)

For coefficient $\Lambda = \mathbb{Z}/\ell^n, \mathbb{Z}_\ell, \mathbb{Q}_\ell$, we have the weight spectral sequence:

$$E_1^{p,q} = \bigoplus_{i\geq 0, i\geq -p} H^{q-2i}(X_{\bar{k}}^{(p+2i)}, \Lambda)(-i) \Rightarrow H^{p+q}(X_{\bar{K}}, \Lambda)$$

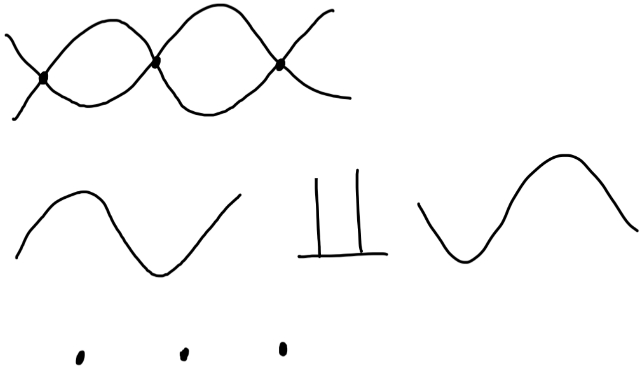
- The inertia group acts on E_1 by $E_1^{p,q} \to E_1^{p+2,q-2}(1)$, and agrees with the action on $H^{p+q}(X_{\bar{K}}, \Lambda)$ after taking graded pieces.
- The differential $d_1^{pq} : E_1^{p,q} \to E_1^{p+1,q}$ are just alternating sums of push-forwards and pull-backs on cohomology.

Weight spectral sequence for a semi-stable curve

$Y = X_{\bar{k}}$

$X^{(0)}$

$X^{(1)}$



$$H^0(X^{(1)}) \xrightarrow{i_*} H^2(X^{(0)}) \longrightarrow 0$$

$$0 \longrightarrow H^1(X^{(0)}) \xrightarrow{N = id} 0$$

$$0 \longrightarrow E_1^{00} = H^0(X^{(0)}) \xrightarrow{i^*} H^0(X^{(1)})$$

Construction

The induced filtration G_i on $H^{p+q}(X_{\bar{K}}, \Lambda)$ satisfies $NG_i \subseteq G_{i-2}$, so it looks like the monodromy filtration. And it is the weight filtration if we know Weil conjecture (and $\Lambda = \mathbb{Q}_\ell$). But firstly, why do we have such spectral sequence? Construction: in any abelian category, we can consider the monodromy filtration of a nilpotent operator on an object, the recipe is the same.
So the nearby cycle $R\Psi\Lambda$ itself has a monodromy filtration, which will give the weight-spectral sequence $E_1^{p,q} = H^{p+q}\left(X_{\bar{k}}, \mathrm{gr}_{-p}^M R\Psi\Lambda\right) \Rightarrow H^{p+q}\left(X_{\bar{K}}, \Lambda\right)$. And $N : \mathrm{gr}_{-p}^M R\Psi\Lambda \to \mathrm{gr}_{-p-2}^M R\Psi\Lambda$ induces the action $N : E_1^{pq} \to E_1^{p+2,q-2}$.

What we need to compute

We only need to compute $\mathrm{gr}^M_{-p} R\Psi\Lambda$, recall the monodromy filtration is the convolution of the kernel filtration $F_i = KerN^i$ and the image filtration $G^j = ImN^j$, and

$$Gr^M_r \cong \bigoplus_{i-j=r} Gr^j_G Gr^F_i.$$

The idea: we have a canonical truncation $F'_i = \tau_{\leq q} R\Psi\Lambda$ by degree. By definition, we just need to compute $i^* R^i j_* \Lambda$, and it turns out that $F'_i = F_i$. It remains to compute the induced image filtration on $R^i\Psi\Lambda$, which is done by understanding the inertial action on $R^i\Psi\Lambda$. The computation is local, and the vanishing cycle will only support on the singular locus.

Curves

If $X \to R$ is just a curve with strictly semi-stable reduction, the local geometry looks like $\mathbb{F}_p[[t]][x,y]/(xy-t) \to \mathbb{F}_p[[t]]$, the special fiber is $xy=0$. Then the computation is easy, and essentially a special case of the Picard-Lefschetz formula in $SGA7$.

Let G be the dual graph of $X_{\bar{s}}$, i.e vertices Σ_0 correspond to the irreducible components of $X_{\bar{s}}$, edges $\Sigma_1 = \Sigma$ correspond to the intersection points of irreducible components of $X_{\bar{s}}$. Then

Picard-Lefschetz formula (in the curve case)

- For any point $x \in \Sigma$, there exists $\delta_x \in H^1(X_{\bar{\eta}})$ well defined up to sign, called the vanishing cycle at x. we have the exact sequence
  $0 \to H^1(X_{\bar{s}}) \xrightarrow{\mathrm{sp}} H^1(X_{\bar{\eta}}) \xrightarrow{(-,\delta_x)} \oplus_{x\in\Sigma}\Lambda(-1) \to H^2(X_{\bar{s}}) \to H^2(X_{\bar{\eta}}) \to 0$
  Here $(a,b) = \mathrm{Tr}(ab)$ with $\mathrm{Tr} : H^2(X_{\bar{\eta}}) \cong \Lambda(-1)$, $(\delta_x, \delta_y) = 0$ if $x \neq y$, $(\delta_x, \delta_x) = 0$.

Curves

If $X \to R$ is just a curve with strictly semi-stable reduction, the local geometry looks like $\mathbb{F}_p[[t]][x,y]/(xy-t) \to \mathbb{F}_p[[t]]$, the special fiber is $xy = 0$. Then the computation is easy, and essentially a special case of the Picard-Lefschetz formula in $SGA7$.

Let G be the dual graph of $X_{\bar{s}}$, i.e vertices Σ_0 correspond to the irreducible components of $X_{\bar{s}}$, edges $\Sigma_1 = \Sigma$ correspond to the intersection points of irreducible components of $X_{\bar{s}}$. Then

Picard-Lefschetz formula (in the curve case)

- For any point $x \in \Sigma$, there exists $\delta_x \in H^1(X_{\bar{\eta}})$ well defined up to sign, called the vanishing cycle at x. we have the exact sequence
  $0 \to H^1(X_{\bar{s}}) \stackrel{\mathrm{sp}}{\to} H^1(X_{\bar{\eta}}) \stackrel{(-,\delta_x)}{\to} \oplus_{x\in\Sigma}\Lambda(-1) \to H^2(X_{\bar{s}}) \to H^2(X_{\bar{\eta}}) \to 0$
  Here $(a,b) = \mathrm{Tr}(ab)$ with $\mathrm{Tr} : H^2(X_{\bar{\eta}}) \cong \Lambda(-1)$, $(\delta_x, \delta_y) = 0$ if $x \neq y$, $(\delta_x, \delta_x) = 0$.

The five-term exact sequence is from the (coarser) spectral sequence $E_2^{pq} = H^p(X_{\bar{s}}, R^q\Psi\Lambda) \Rightarrow H^{p+q}(X_{\bar{\eta}})$. And $H^1(X_{\bar{s}}) = H^1(G, \Lambda)$ (cohomology of a graph), $H^0(X_{\bar{s}}, R^1\Psi\Lambda) = \oplus_{x\in\Sigma_1}\Lambda(-1)$, $H^2(X_{\bar{s}}) = \oplus_{x\in\Sigma_0}\Lambda(-1)$ (top degree etale cohomology counts the number of irreducible components).

General results

Let $Y = X_{\bar{s}}$, $a_i : Y^{(i)} \to Y$ the union of closed immersions $Y_I \to Y$.

Prop 1.3 in Saito's paper

Corollary 1.1.3 *1. Let $\delta : \Lambda_Y \to a_{0*}\Lambda$ be the canonical map. Then, we have an isomorphism*

$$\begin{array}{ccccccccc}
0 \to \Lambda_Y & \xrightarrow{\delta} & a_{0*}\Lambda & \overset{\delta\wedge}{\to} \cdots \overset{\delta\wedge}{\to} & a_{p*}\Lambda & \overset{\delta\wedge}{\to} \cdots \overset{\delta\wedge}{\to} & a_{n*}\Lambda & \to 0 \\
\| & & \downarrow \theta' & & \downarrow \theta' & & \downarrow \theta' & \\
0 \to \Lambda_Y & \xrightarrow[\theta]{} & i^*R^1 j_*\Lambda(1) & \underset{\theta\cup}{\to} \cdots \underset{\theta\cup}{\to} & i^*R^{p+1} j_*\Lambda(p+1) & \underset{\theta\cup}{\to} \cdots \underset{\theta\cup}{\to} & i^*R^{n+1} j_*\Lambda(n+1) & \to 0
\end{array}$$

of exact sequences.

2. For $p \geq 0$, we have an exact sequence

$$0 \longrightarrow R^p\psi\Lambda \xrightarrow{\bar{\theta}} i^*R^{p+1}j_*\Lambda(1) \xrightarrow{\theta\cup} \ldots \xrightarrow{\theta\cup} i^*R^{n+1}j_*\Lambda(n+1-p) \longrightarrow 0$$

Part 1: Here θ' are all isomorphisms, $R^p\Psi\Lambda = \mathrm{Ker}(a_{p*}\Lambda(-p) \to a_{p+1*}\Lambda(-p))$, $(R^1\Psi\Lambda) = CoKer(\Lambda_Y \to \oplus_{i=1}^m \Lambda_{Y_i}) \otimes \Lambda(-1)$, $R^q\Psi\Lambda = \wedge^q R^1\Psi\Lambda$.

Part 2:$R\Gamma(I_\ell, R\Psi\Lambda) = i^*Rj_*\Lambda$, I_ℓ is cyclic, and I_ℓ acts trivially on $R^q\Psi\Lambda$ (by computation, which is **a feature for the semi-stable reduction**), so we have a short exact sequence $0 \to R^n\Psi\Lambda \to i^*R^nj_*\Lambda \to R^{n+1}\Psi\Lambda \to 0$, the second arrow is the $\bar{\theta}$.

A local-global computation

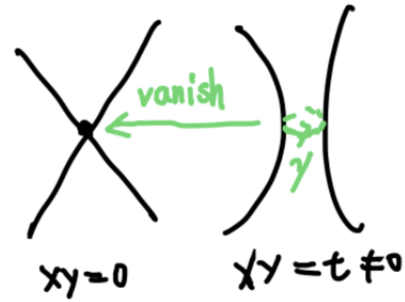
The computation will use the absolute purity conjecture (proved by Gabber and Fujiwara), but I think it can be avoid in equal characteristic.
By an easy combinatoric exercise, $\Lambda_Y \to a_{0*}\Lambda \to a_{1*}\Lambda \to ... \to a_{n*}\Lambda$ is exact (here we take alternating sums of i_*), the non-trivial thing is to show $i^* R^p j_* \Lambda = a_{p*}\Lambda(-p)$.
Curve case: $\mathbb{F}_p[[t]][x, y]/(xy - t) \to \mathbb{F}_p[[t]]$ globalizes to $\mathbb{F}_p[t][x, y]/(xy - t) \to \mathbb{F}_p[t]$, so the total space is $\mathbb{A}^2$, the complement of the special fiber is $j^2 : \mathbb{A}^2 - \{xy = 0\} = (\mathbb{A}^1 - 0)^2 \hookrightarrow \mathbb{A}^2$.
(**we ignore the base!**)

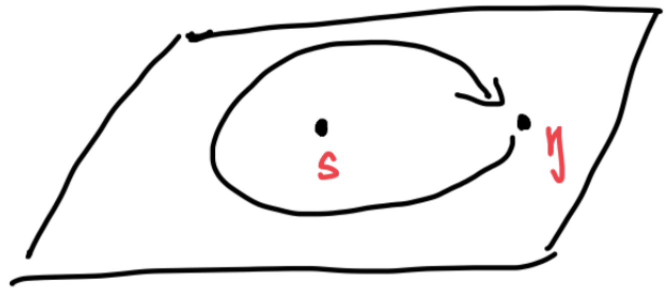
A local-global computation

The computation will use the absolute purity conjecture (proved by Gabber and Fujiwara), but I think it can be avoid in equal characteristic.
By an easy combinatoric exercise, $\Lambda_Y \to a_{0*}\Lambda \to a_{1*}\Lambda \to ... \to a_{n*}\Lambda$ is exact (here we take alternating sums of i_*), the non-trivial thing is to show $i^* R^p j_* \Lambda = a_{p*}\Lambda(-p)$.
Curve case: $\mathbb{F}_p[[t]][x, y]/(xy - t) \to \mathbb{F}_p[[t]]$ globalizes to $\mathbb{F}_p[t][x, y]/(xy - t) \to \mathbb{F}_p[t]$, so the total space is $\mathbb{A}^2$, the complement of the special fiber is $j^2 : \mathbb{A}^2 - \{xy = 0\} = (\mathbb{A}^1 - 0)^2 \hookrightarrow \mathbb{A}^2$.
(**we ignore the base!**)
Let $j : \mathbb{A}^1 - \{0\} \hookrightarrow \mathbb{A}^1, i : \{0\} \hookrightarrow \mathbb{A}^1$, what is $i^* R^k j_* \Lambda$? By definition, it's the stalk of $R^k j_* \Lambda$ at 0, i.e $\mathrm{colim}_{U \to \mathbb{A}^1 \text{"open"}} H^k(U - 0, \Lambda)$, it's $\Lambda(-1)$ if $k = 1$, and Λ if $k = 0$, and zero else (image you are computing cohomology of $\mathbb{C} - 0$). To compute the nearby cycle i.e $\bar{i}^* R\bar{j}_* \Lambda$, go up along the Kummer tower (adding $t^{1/n}$), the computation is similar.
Let $j^n : (\mathbb{A}^1 - \{0\})^n \hookrightarrow (\mathbb{A}^1)^n$, $n = 2$ is the curve case. By Kunneth formula, we computes nearby cycles for all n, hence the proposition.

A picture

$$\mathbb{C}^2 \to \mathbb{C}, \quad (x,y) \mapsto xy$$





$\gamma \in H_1(X_{\bar{\eta}})$ is 0 in $H_1(X_{\bar{s}})$

The monodromy filtration on the nearby cycle

So $R^i\psi\Lambda[-i]$ is quasi-isomorphic to $[a_{i*}\Lambda(-i) \to ... \to a_{n*}\Lambda(-i)]$ (the degree is from i to n). Then by computing the inertia action on $R\psi\Lambda$, one see the decreasing image filtration G^j on $R^i\psi\Lambda[-i] = Gr_i^F R\psi\lambda$ is the same as the truncation by $[a_{i+j*}\Lambda(-i) \to ... \to a_{n*}\Lambda(-i)]$
We finally get
$Gr_r^M R\psi\Lambda \cong \bigoplus_{i-j=r} Gr_G^j R^i\psi\Lambda \cong \bigoplus_{i-j=r} a_{(i+j)*}\Lambda(-i)[-(i+j)]$.
Now we have the weight spectral sequence, and we return to the proof of Weil conjecture.

Weights and monodromy

If you have a Galois representation $\mathrm{Gal}_{\mathbb{Q}} \to GL_n(\mathbb{Q}_\ell)$ coming from geometry, it is unramfied almost everywhere. If you know that there exists an integer k such that for all but finitely many unramified p, all eigenvalues of $Frob_p$ have absolute value $p^{k/2}$.

Then by density theorem, you may believe all eigenvalues of $Frob_p$ have absolute value $p^{k/2}$ for all unramified p. Even for ramified p, we can't choose a canonial $Frob_p$, but it's reasonable to believe all eigenvalues of $Frob_p$ on V^{I_p} has weight no bigger than k.

Plus some linear algebras on tensor product and duality, one may believe the monodromy filtration has pure graded pieces.

Weights and monodromy

If you have a Galois representation $\mathrm{Gal}_{\mathbb{Q}} \to GL_n(\mathbb{Q}_\ell)$ coming from geometry, it is unramfied almost everywhere. If you know that there exists an integer k such that for all but finitely many unramified p, all eigenvalues of $Frob_p$ have absolute value $p^{k/2}$.
Then by density theorem, you may believe all eigenvalues of $Frob_p$ have absolute value $p^{k/2}$ for all unramified p. Even for ramified p, we can't choose a canonial $Frob_p$, but it's reasonable to believe all eigenvalues of $Frob_p$ on V^{I_p} has weight no bigger than k.
Plus some linear algebras on tensor product and duality, one may believe the monodromy filtration has pure graded pieces.
In equal characteristic, one can easily prove this is indeed true using L-function and Rankin-method, if we can globalize the DVR to a curve, and we know other fibers have pure cohomology. This is section 1.6-1.8 of Weil II.

The set up

T is a smooth affine curve over k, and we have a projective and strictly semi-stable family $f : E \to T$.
So there is a closed point $t \in |T|$, such that over $U = T - \{t\}$ $f : E_U \to U$ is smooth, each fiber over U satisfies the purity assumption (by the proof of Weil conjecture for smooth projective hypersurfaces). Ant $E_t = f^{-1}(t)$ has a generic finite dominant map to our starting X_0. We only need to show the $H^i(E_{\bar{t}})$ has weights no bigger than i.

The End

Now we apply weight spectral sequence to E_R over $R = \hat{O_{T,t}}$.

$$E_1^{p,q} = \bigoplus_{i \geq 0, i \geq -p} H^{q-2i}(X_{\bar{k}}^{(p+2i)}, \Lambda)(-i) \Rightarrow H^{p+q}(X_{\bar{K}}, \Lambda).$$

By induction of dimension and weak Lefschetz, we see $E_1^{p,q}$ is pure unless $p + 2i = 0, q - 2i = n$, but $i \geq 0, i \geq -p$ shows $i = 0, p = 0, q = n$. Set $V = H^n(X_{\bar{K}}, \mathbb{Q}_\ell)$, which satisfies the weight-monodromy conjecture. So the graded piece $gr_k^G V$ is pure unless $k = 0$.

By design, $NG_k \subseteq G_{k-2}$. Now we use lemma 2.6 in Scholl's paper:

The End

Now we apply weight spectral sequence to E_R over $R = \hat{O_{T,t}}$.

$$E_1^{p,q} = \bigoplus_{i \geq 0, i \geq -p} H^{q-2i}(X_{\bar{k}}^{(p+2i)}, \Lambda)(-i) \Rightarrow H^{p+q}(X_{\bar{K}}, \Lambda).$$

By induction of dimension and weak Lefschetz, we see $E_1^{p,q}$ is pure unless $p + 2i = 0, q - 2i = n$, but $i \geq 0, i \geq -p$ shows $i = 0, p = 0, q = n$. Set $V = H^n(X_{\bar{K}}, \mathbb{Q}_\ell)$, which satisfies the weight-monodromy conjecture. So the graded piece $gr_k^G V$ is pure unless $k = 0$.

By design, $NG_k \subseteq G_{k-2}$. Now we use lemma 2.6 in Scholl's paper:

Lemma

Let V be monodromy-pure of weight n, and let G_* be a filtration on V by G_K-invariant subspaces such that

- $NG_k \subseteq G_{k-2}$ *for all* k.
- *The graded piece* $gr_k^G V$ *is pure of weight* $k + n$, *for all* $k \neq 0$.

Then G_* is the monodromy filtration, hence $gr_0^G V$ is pure of weight n.

We see $H^n(X^{(0)})$ has weight no bigger than n, hence $H^n(X_{\bar{s}})$ has weight no bigger than n by excision, we're done.

References

- Yifeng Liu, Bounding cubic-triple product Selmer groups of elliptic curves, Journal of the European Mathematical Society, 21 (2019) 1411-1508
- Luc Illusie, Glimpses on vanishing cycles, from Riemann to today, avaliable at `https://www.imo.universite-paris-saclay.fr/~illusie/Illusie-Sanya3.pdf`
- Weizhe Zheng, Nearby cycles over general bases, avaliable at `https://server.mcm.ac.cn/~zheng/slides/nearby-slides.pdf`.
- Tony Scholl, Hypersurfaces and the Weil conjectures, IMRN 2010.
- Luc Illusie, On semistable reduction and the calculation of nearby cycles, preprint, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter, Berlin, 2004, pp. 785–803.
- Takeshi Saito, Weight spectral sequences and independence of ℓ, Journal of the Institute of Mathematics of Jussieu, 2003.
- Grothendieck-Serre correspondence, avaliable at `https://webusers.imj-prg.fr/~leila.schneps/grothendieckcircle/Letters/GS.pdf`.

Thank you! Questions?