The Riemann Hypothesis for Hypersurfaces

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- First of all, we recall the cohomology of $\mathbb{P}^n$: $\mathrm{H}^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/h^{n+1}$ where H is the class of a hyperplane H.

Lefschetz Hyperplane Theorem

Theorem (Lefschetz Hyperplane Theorem)

Let X be a smooth ample divisor of a smooth projective variety Y with $\dim Y = n+1$. Suppose the base field k is algebraically closed. Write $\mathrm{H}^j(-)$ for $\mathrm{H}^j_{\acute{e}t}((-), \mathbb{Q}_\ell)$, $\ell \neq \mathrm{char}\, \mathsf{k}$.

1. *$\mathrm{H}^j(Y) \to \mathrm{H}^j(X)$ is bijective for $j < n$ and injective for $j = n$.*
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- Apply the above to $Y := \mathbb{P}^{n+1}$. Again let h denote the class of a hyperplane. For all $j \neq n$, we have $\mathrm{H}^j(X) \cong \mathbb{Q}_\ell(h^j) \cong \mathbb{Q}_\ell(j)$.

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- The only interesting part of the cohomology of X is $H^n(X)$.

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- Let $h \in \mathrm{H}^2(X)$ be the hyperplane section. Consider the primitive cohomology

$$\mathrm{P}^n(X) = \mathrm{H}^n(X) \text{ if } n \text{ is odd, and } \mathrm{P}^n(X) = \mathrm{H}^n(X)/\langle h^{n/2} \rangle \text{ if } n \text{ is even.}$$

Then we have $H^*(X) \cong H^*(\mathbb{P}^n) \oplus \mathrm{P}^n(X)$.

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  Then we have $H^*(X) \cong H^*(\mathbb{P}^n) \oplus \mathrm{P}^n(X)$.
- The Zeta function $Z(X_0, T)$ is given by

  $$\frac{P(T)}{\prod_{i=0}^{n}(1 - q^i T)} (n \text{ odd}), \text{ or } \frac{1}{P(T)\prod_{i=0}^{n}(1 - q^i T)} (n \text{ even})$$

  with $P(T) := \det(1 - T\mathrm{Frob}_q|_{\mathrm{P}^n(X_0)})$.

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  with $P(T) := \det(1 - T\mathrm{Frob}_q|_{\mathrm{P}^n(X_0)})$.
- In either case, we see that $P(T)$ has integer coefficients. In particular, the coefficients are totally real.

Reduction to Point Counting

- Since $\alpha \mapsto q^n/\alpha$ defines an involution on the eigenvalues of Frob_q, it suffices to show that every eigenvalue of Frob_q on $\mathrm{P}^n(X)$ has ι-norm $\leq q^{n/2}$, for any $\iota : \bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$.

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- Recall that $H^*(X) \cong H^*(\mathbb{P}^n) \oplus \mathrm{P}^n(X)$, so that

$$\#X_0(\mathbb{F}_{q^r}) = \#\mathbb{P}^n(\mathbb{F}_{q^r}) + (-1)^r \mathrm{tr}((\mathrm{Frob}_q)^r|_{\mathrm{P}^n(X)})$$

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  by the point counting formula.
- It suffices to show that

$$\#X_0(\mathbb{F}_{q^r}) = \#\mathbb{P}^n(\mathbb{F}_{q^r}) + O(q^{rn/2})$$

  as $r \geq 1$ varies.

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- Let Frob_u be the generator of $\mathrm{Gal}_{\mathsf{k}(u)}$ given by $\lambda \mapsto \lambda^{\#\mathsf{k}(u)}$. Set

$$P_{\mathcal{F},u}(T) := \det(1 - T^{[\mathsf{k}(u):k]}\mathrm{Frob}_u|_{\mathcal{F}_{\bar{u}}})$$

and

$$L_{\mathcal{F}}(T) := \prod_{u \in U} P_{\mathcal{F},u}(T)^{-1}.$$

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Theorem

Let $\mathcal{F}$ be an ℓ-adic local system on U_0 which is ι-real. Suppose that for some closed point $u \in U_0$, every eigenvalue $\alpha_{i,u}$ of $\mathrm{Frob}_u|_{\mathcal{F}_{\bar{u}}}$ satisfies $|\iota(\alpha_{i,u})| \leq 1$. Then the same is true for any other closed point $u' \in U_0$.

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- Given two homogenous polynomials F_0, F_1, we can consider the pencil $tF_0 + (1-t)F_1$, thereby obtaining a family over $\mathbb{A}^1$. If F_0, F_1 are smooth, by removing singular fibers, we obtain a family $f : \mathcal{X} \to U$ containing F_0, F_1.

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- Apply the proposition to $\mathcal{F} := \mathbb{R}^n f_* \bar{\mathbb{Q}}_\ell(n/2)$. (Recall that the Zeta function ensures that our $\mathcal{F}$ is in fact integral.)

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- If $\operatorname{Spec} \mathcal{O}_G$ is equipped with the structure of an algebraic group, then $\mathcal{O}_G$ is equipped with a distinguished $\mathcal{O}_G \to k$ and a co-multiplication structure $\Delta : \mathcal{O}_G \to \mathcal{O}_G \otimes \mathcal{O}_G$.

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- For example, $\mathbb{G}_m$ is given by $\mathcal{O}_G := k[T^{\pm}]$ and $\Delta(T) = T \otimes T$.
- A *character* of a group G is a morphism $G \to \mathbb{G}_m$. The group of characters is denoted by $X^*(G)$.

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- Given a finitely generated abelian group M. Consider the functor $\mathrm{Hom}(M, \mathbb{G}_m) : \mathsf{Alg}/k \to \mathsf{Grp}$ defined by $R \mapsto \mathrm{Hom}(M, R^\times)$.

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- A morphism $G \to Q$ of algebraic groups is called a quotient map if it is *faithfully flat*.
- Fact: Every normal algebraic group N of G arises the kernel of a quotient map $G \to Q$.

Interlude: Galois cohomology

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- Hilbert theorem 90: $\mathrm{H}^1(k, \mathbb{G}_m(\bar{k})) = 0$ for any field k.
- Example: $\mathrm{H}^1(k, \mu_n(\bar{k})) = k^\times/(k^\times)^n$, for $\mu_n := \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$.

Interlude : Gauss Sums

- Let $\mathbb{F}_q$ be a finite field. $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a mutliplicative character and $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ be an additive character.

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- The proof uses orthogonality relaitions in the character theory of finite groups, as well as some elementary tricks of rearranging the order of summations.

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- Let $\mathbb{F}_q$ be a finite field. $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a mutliplicative character and $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ be an additive character.
- Set $\mu(\chi, \psi) := \sum_{x \in \mathbb{F}_q^\times} \chi(x)\psi(x)$.
- It is hard to compute $\mu(\chi, \psi)$ in general, but we know its norm:

Theorem

If χ and ψ are both nontrivial, then $|\mu(\chi, \psi)| = \sqrt{q}$.

- The proof uses orthogonality relaitions in the character theory of finite groups, as well as some elementary tricks of rearranging the order of summations.
- Reference: Kowalski's notes `https://people.math.ethz.ch/~kowalski/exp-sums.pdf`.

Point Counting

- Recall that it suffices to show that for some smooth hypersurface X_0 of degree d in $\mathbb{P}^{n+1}$, we have the point-count estimate:

$$\#X_0(\mathbb{F}_{q^r}) = \#\mathbb{P}^n(\mathbb{F}_{q^r}) + O(q^{rn/2})$$

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$$\mathrm{H}^n(X) = \begin{cases} \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

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- To compute $\dim \mathrm{H}^n(X)$, it suffices to compute the Euler characteristic. Moreover, when n is even, $\mathrm{H}^n(X)$ is spanned by algebraic classes. (e.g., $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ when $n = 2$.)

Gabber's Hypersurface

- If $d \geq 3$ and $p \mid d$, use Gabber's hypersurface

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- For an N-tuple $W = (w_1, \cdots, w_N)$ of nonnegative integers, we write X^W for the monomial $X_1^{w_1} \cdots X_N^{w_N}$.

Theorem

Let $N \geq 1$, and $X^{W_1}, \cdots, X^{W_N}$ be N monomials in N variables with W_i's linearly independent over $\mathbb{Q}$. Suppose that each variable X_i occurs in at most two of these monomials. Then for $V := \sum_i X^{W_i} = 0$ in $\mathbb{A}^N$, we have $\# V(\mathbb{F}_q) = q^{N-1} + O(q^{N/2})$.

Delsarte's Theorem

Theorem (Delsarte)

Let $N > k \geq 0$ and suppose given $N-k$ linearly independent monomials $X^{W_1}, \cdots, X_N^{W_N}$ in N variables. Let $V := \{\sum_i X_i^{W_i} = 0\}$ and $V^* := V \cap (\mathbb{G}_m^N \cap \mathbb{A}_m^N)$. Then $\#V^*(\mathbb{F}_q) = q^{-1}(q-1)^N + O(q^{(N+k)/2})$.

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- We view the $N-k$ linearly independent vectors W_i as giving rise to an surjection $\phi : \mathbb{G}_m^N \to \mathbb{G}_m^{N-k}$ of split tori over $\mathbb{F}_q$: $(X_1, \cdots, X_N) \mapsto (X^{W_1}, \cdots, X^{W_{N-k}})$. Then we only need to show:

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$$\#\{x \in \mathbb{G}_m^N(\mathbb{F}_q) : \Sigma(\phi(x)) = 0\} = \frac{(q-1)^N}{q} + O(q^{(N+k)/2}).$$

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- We first treat the free part.

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- Recall that we want a big O estimate of the LHS, now we reduce to considering the RHS. Hence we reduce to the case $k = 0$.

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- Our new goal: Given $0 \to \mu_{M_{\mathrm{tor}}} \to \mathbb{G}_m^N \xrightarrow{\phi} \mathbb{G}_m^N \to 0$, estimate $T := \#\{x \in \mathbb{G}_m^N(\mathbb{F}_q)(\mathbb{F}_q) : \Sigma(\phi(x)) = 0\}$. Let's write $\mu_{M_{\mathrm{tor}}}$ simply as μ.

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- Therefore, we get

$$T = \sum_{t \in \mathbb{G}_m^N(\mathbb{F}_q), \sum_i t_i = 0} \sum_{\chi \in \mathcal{K}} \chi(t).$$

Gauss Sums

- Take a nontrivial additive character $\psi : \mathbb{G}_a(\mathbb{F}_q) = \mathbb{F}_q \to \mathbb{C}^\times$. We have

$$\sum_{a \in \mathbb{F}_q} \psi(ax) = q \text{ if } x = 0 \text{ and } 0 \text{ otherwise.}$$

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- Now we rewrite the sum as

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- Recall that we have

Theorem (Modulus of Gauss sums)

For any nontrivial additive character ψ and nontrivial multiplicative character χ on $\mathbb{F}_q$, $|\sum_{t \in \mathbb{F}_q^\times} \chi(t)\psi(t)| = \sqrt{q}$.

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- To sum up, we have

$$|T - \frac{(q-1)^N}{q}| \leq \frac{q-1}{q} \# M_{\mathrm{tor}} q^{N/2}.$$

Gauss Sums

$$T = \frac{1}{q} \sum_{a \in \mathbb{F}_q} \sum_{\chi \in \mathcal{K}} \sum_{t \in \mathbb{G}_m^N(\mathbb{F}_q)} \chi(t) \psi(a \sum_i t_i)$$

- The $a = 0$ term is $q^{-1} \sum_{\chi \in \mathcal{K}} \sum_{t \in \mathbb{G}_m^N(\mathbb{F}_q)} \chi(t) = q^{-1}(q-1)^N$.
- For the other a's, the term has norm $q^{N/2}$ (a product of N Gauss sums). There are $\#\mathcal{K} = \#\mathrm{H}^1(\mathbb{F}_q, \mu(\bar{\mathbb{F}}_q)) \leq \#M_{\mathrm{tor}}$ of them.
- To sum up, we have

$$|T - \frac{(q-1)^N}{q}| \leq \frac{q-1}{q} \#M_{\mathrm{tor}} q^{N/2}.$$

- Now we are done!

(From now on optional) Torsors

- Let G be a group and S be a set on which G acts on the right. We say that S is a G-torsor if for every $s \in S$, the map $G \longrightarrow S$ defined by $g \mapsto sg$ is a bijection.

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- Torsors of a finite group G over X are representable by a Galois cover of X (not necessarily connected).
- If G is a profinite abelian group, then there is a canonical identification $\mathrm{H}^1(X_{\text{ét}}, G) = \mathrm{Hom}_{\text{cts}}(\pi_1^{\text{ét}}(X), G)$.

Persistence of purity

- We go over the persistence of purity theorem with a bit more detail.

Theorem (Persistence of Purity)

Let $\mathcal{F}$ be an ℓ-adic local system on U_0 which is ι-real. Suppose that for some closed point $u_0 \in U_0$, every eigenvalue α_{i,u_0} of $\mathrm{Frob}_{u_0}|_{\mathcal{F}_{\bar{u}_0}}$ satisfies $|\iota(\alpha_{i,u_0})| = 1$. Then the same is true for any other closed point $u \in U_0$.

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Lemma

Let $\mathcal{L}$ be an ℓ-adic local system on U_0 of rank 1. Then for some power $\mathcal{L}^{\otimes m}$, there exists $\alpha \in \bar{\mathbb{Q}}_\ell^\times$ such that $\mathrm{Frob}_u|_{\mathcal{L}^{\otimes m}} = \alpha^{\deg u}$.

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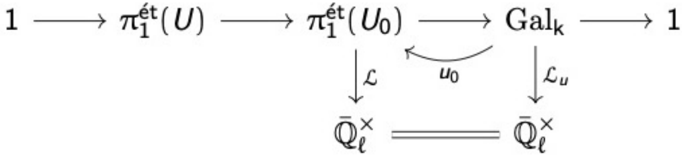

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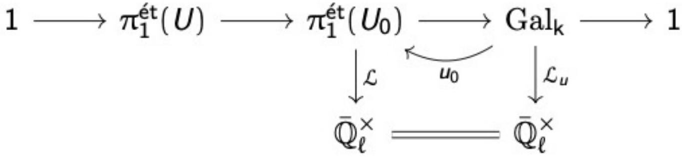

- Via the splitting, $\mathrm{Gal}_{\mathsf{k}}$ acts by conjugation on $\pi_1^{\text{ét}}(U)$, and hence on $\pi_1^{\text{ét}}(U)$-representations. This action fixes $\mathcal{L}|_{\pi_1^{\text{ét}}(U)}$.

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- Therefore, $\mathcal{L}|_{\pi_1^{\text{ét}}(U)}$ is trivial.

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- The point is that the second isomorphism does not depend on the choice of the path $\bar{u}_0 \rightsquigarrow \bar{u}$.

Proof of Persistence of Purity

- Suppose we have another k-point $u \in U$. Then we have two sections.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1^{\text{ét}}(U)^{\mathrm{ab}}/? & \longrightarrow & \pi_1^{\text{ét}}(U_0)^{\mathrm{ab}} & \underset{u_0}{\overset{u}{\rightleftarrows}} & \mathrm{Gal}_{\mathsf{k}} & \longrightarrow & 1 \\ & & & & \downarrow \mathcal{L} & & \downarrow \mathcal{L}_u \ \downarrow \mathcal{L}_{u_0} & & \\ & & & & \bar{\mathbb{Q}}_\ell^\times & = & \bar{\mathbb{Q}}_\ell^\times & & \end{array}$$

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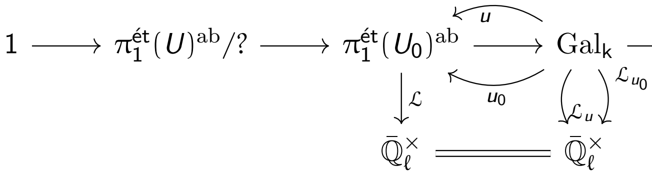

- Now the representation $\mathcal{L}_{u_0} - \mathcal{L}_u : \mathrm{Gal}_{\mathsf{k}} \to \bar{\mathbb{Q}}_\ell^\times$ is induced by

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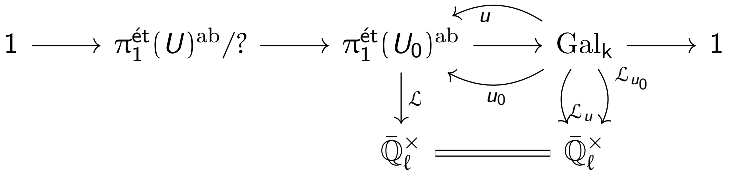
Proof of Persistence of Purity

- Suppose we have another k-point $u \in U$. Then we have two sections.



- Now the representation $\mathcal{L}_{u_0} - \mathcal{L}_u : \mathrm{Gal}_{\mathsf{k}} \to \bar{\mathbb{Q}}_\ell^\times$ is induced by

$$\mathrm{Gal}_{\mathsf{k}} \stackrel{u_0 - u}{\longrightarrow} \pi_1^{\text{ét}}(U_0)^{\mathrm{ab}} \to \bar{\mathbb{Q}}_\ell^\times.$$

- Since $u_0 - u$ lands in $\pi_1^{\text{ét}}(U)^{\mathrm{ab}}/?$, the above composition vanishes.