The Hitchhiker's Guide to Crystalline Cohomology

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Motivation: de Rham cohomology of liftings

- Suppose $X/\operatorname{Spec} k$ has multiple smooth lifts Z to $W(k)$. Suprisingly, $H^i_{\mathrm{dR}}(Z/W)$ is independent of the choice.
- Example: let $f : \mathcal{X} \to S = \operatorname{Spf} W[\![t]\!]$ lift X/k.
- Let $\mathcal{H}^i$ be relative de Rham cohomology. In good situations,

$$H^i_{\mathrm{dR}}(Z/W) = \mathcal{H}^i \otimes_{W[\![t]\!]} (W, x)$$

where $x : W[\![t]\!] \to W$ is a specialization.

Independence of liftings

- The Gauss-Manin connection $\nabla : \mathcal{H}^i \to \mathcal{H}^i \otimes \Omega^1_{S/W}$ identifies these fibers.
- An explicit isomorphism $\mathcal{H}^i \otimes_{W[\![t]\!]} (W, x) \to \mathcal{H}^i \otimes_{W[\![t]\!]} (W, y)$ can be defined by:

$$s \mapsto \sum_{n \geq 0} \frac{(x(t) - y(t))^n}{n!} \left(\nabla \left(\frac{\partial}{\partial t} \right)^n s \right) (y)$$

- Grothendieck thought that there should be a $W(k)$-valued cohomology intrinsic to X/k.

Divided power structures

Let A be a ring and $I \subset A$ be an ideal. A PD-structure on I is a series of maps $\gamma_n : I \to A$ for $n \geq 0$ so that:

- $\gamma_0(x) = 1$ and $\gamma_1(x) = x$
- $\gamma_n(x) \in I$ if $n \geq 1$
- $\gamma_n(x + y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(y)$
- $\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$
- $\gamma_n(x)\gamma_m(x) = \binom{m+n}{n}\gamma_{n+m}(x)$
- $\gamma_m(\gamma_n(x)) = \frac{(mn)!}{m!(n!)^m}\gamma_{mn}(x)$

Check: $\gamma_n(x) = x^n/n!$ works if A is a $\mathbb{Q}$-algebra.

PD-structures, continued

Why divided powers?

- It's exactly what you need to integrate n times.

Observation: $n!\gamma_n(x) = x^n$

- $\gamma_0(x) = 1$
- $\gamma_1(x)\gamma_n(x) = \binom{n+1}{1}\gamma_{n+1}(x) = (n+1)\gamma_{n+1}(x)$

Some consequences:

- If $A = W(\mathbb{F})$, then there is a unique PD-structure on (p) given by $\gamma_n(p) = p^n/n!$. This induces a canonical PD-structure on W/p^k, and on $pA \subset A$ for any W-algebra A.
- If A is killed by n, then I is a nil-ideal: $x^n = 0$ for all $x \in I$.

PD-structures: example

Let A be any ring. There is a "universal" PD-structure in r variables, $A\langle x_1, \ldots, x_r \rangle$.

- Generated as an A-module by symbols $x_1^{[k_1]} \cdots x_r^{[k_r]}$
- PD-structure will satisfy

$$\gamma_n(x_i^{[1]}) = x_i^{[n]}, \text{ so that morally: } x_i^{[n]} = \frac{x_i^n}{n!}$$

- Multiplication defined by:

$$x_i^{[m]} x_i^{[n]} = \binom{m+n}{n} x_i^{[m+n]}$$

- Graded ring structure $\oplus \Gamma^n$, where Γ^n is generated by symbols with $k_1 + \ldots + k_r = n$. The ideal is $I = \oplus_{n \geq 1} \Gamma^n$

Crystalline Poincaré lemma (simple version)

Observation: $t_i^{[k]} \mapsto t_i^{[k-1]} dt_i$ is an integrable (square-zero) connection on $A\langle t_1, \ldots, t_n \rangle$ as a module over $A[t_1, \ldots, t_n]$.

Lemma 1

Let A be any ring. The de Rham complex over $A[t_1, \ldots, t_n]$ with values in $A\langle t_1, \ldots, t_n \rangle$ is a resolution of A.

Proof.

We will show the case $n = 1$. This is the complex $A\langle t \rangle \to A\langle t \rangle dt$ given by the map

$$\sum_{k \geq 0} a_k t^{[k]} \mapsto \sum_{k \geq 1} a_k t^{[k-1]} dt.$$

So, $0 \to A \to A\langle t \rangle \to A\langle t \rangle dt \to 0$ is exact. □

PD-structures, continued

- If (A, I, γ) and (B, J, δ) are PD-rings, then a map $\phi : A \to B$ is a PD-morphism if:
  - $\phi(I) \subset J$
  - $\phi(\gamma_n(x)) = \delta_n(\phi(x))$

$$\begin{array}{ccc} \operatorname{Spec} B_0 & \longrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow \phi \\ \operatorname{Spec} A_0 & \longrightarrow & \operatorname{Spec} A \end{array}$$

- If $Z \hookrightarrow X$ is a closed immersion of schemes, have a notion of PD-structure on $\mathcal{I}_Z \subset \mathcal{O}_X$. Sometimes, (Z, X, γ) or $(X, \mathcal{I}_Z, \gamma)$ is called a PD-scheme.

Crystalline site: objects

- $k =$ perfect field of char p, $X/\operatorname{Spec} k$ a fixed scheme.
- $W = W(k)$ and $W_n = W/p^n$ with canonical PD-structure.
- Objects of $\mathrm{Cris}(X/W_n)$ are PD-schemes (U, T, δ) where $U \subset X$ is a Zariski open and the following diagram is a PD-morphism (but not necessarily a pullback).

$$\begin{array}{ccc} U & \overset{i}{\hookrightarrow} & T \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} W_n \end{array}$$

- Since $p^n = 0$ on $\mathcal{O}_T$, $U \hookrightarrow T$ is a homeomorphism.

Crystalline site: morphisms

Morphisms in $\mathrm{Cris}(X/W_n)$ from (U', T', δ') to (U, T, δ) are diagrams:

$$\begin{array}{ccc} U' & \overset{i'}{\hookrightarrow} & T' \\ \downarrow & & \downarrow \\ U & \overset{i}{\hookrightarrow} & T \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, k & \longrightarrow & \mathrm{Spec}\, W_n \end{array}$$

compatible with PD-structures. Here, $U' \to U$ is an open immersion.

Crystalline site: examples

- Stupid example: $U = X$ and $T = X$, considered as a scheme over $\operatorname{Spec} W_n$. The PD-structure on $(0) \subset \mathcal{O}_X$ is trivial.

$$\begin{array}{ccc} X & = & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} W_n \end{array}$$

- $U = X = \operatorname{Spec} k$ and $T = \operatorname{Spec} W_n\langle t_1, \ldots, t_r\rangle$

$$\begin{array}{ccc} \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} W_n\langle t_1, \ldots, t_r\rangle \\ \| & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} W_n \end{array}$$

Crystalline site: examples

Suppose X lifts to a smooth scheme $X'/\operatorname{Spec} W_n$. Can we find a PD-structure so that

$$\begin{array}{ccc} X & \hookrightarrow & X' \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} W_n \end{array}$$

is an object of $\operatorname{Cris}(X/W_n)$? Yes, if this is a pullback.

Crystalline site: sheaves

- The covering families are collections of maps $\{(U_i, T_i, \delta_i)\} \to (U, T, \delta)$ such that $T_i \to T$ is an open immersion, U_i and δ_i are induced by pullback, and $T = \cup T_i$.
- What does it mean to be a sheaf on $\mathrm{Cris}(X/W_n)$?
- Given a sheaf $\mathcal{F}$, and an object (U, T, δ), produce $\mathcal{F}_{(U,T,\delta)}$ a Zariski sheaf on T by the rule:

$$\mathcal{F}_{(U,T,\delta)}(W) = \mathcal{F}(U \times_T W, W, \delta)$$

Crystalline site: sheaves

Proposition

The data of a sheaf $\mathcal{F}$ on $\mathrm{Cris}(X/W_n)$ is equivalent to the data of a Zariski sheaf $\mathcal{F}_{(U,T,\delta)}$ on T for each object (U,T,δ), with maps $g^*_{\mathcal{F}} : g^{-1}\mathcal{F}_{(U',T',\delta')} \to \mathcal{F}_{(U,T,\delta)}$ for each morphism $(U,T,\delta) \xrightarrow{g} (U',T',\delta')$, satisfying:

- *Transitivity for $(U,T,\delta) \to (U',T',\delta') \to (U'',T'',\delta'')$.*
- *If $T \to T'$ is an open immersion and U, δ are induced by pullback, then $g^*_{\mathcal{F}}$ is an isomorphism.*

Proof of proposition

- Suppose given the data of $\mathcal{F}_{(U,T,\delta)}$. Define $\mathcal{F}(U, T, \delta) = \mathcal{F}_{(U,T,\delta)}(T)$.
- Transition maps: suppose $g : (U, T, \delta) \to (U', T', \delta')$ is a morphism.
  - We need a map $\mathcal{F}(U', T', \delta') \to \mathcal{F}(U, T, \delta)$.
  - Note that $\mathcal{F}(U', T', \delta')$ has a map to

$$g^{-1}\mathcal{F}_{(U',T',\delta')}(T) = \varinjlim_{\mathrm{im}(T)\subset W'} \mathcal{F}_{(U',T',\delta')}(W').$$

  - This maps to $\mathcal{F}_{(U,T,\delta)}(T)$ via $g^*_{\mathcal{F}}$.
- Cocycle property of transition maps follows from that of $g^*_{\mathcal{F}}$.

Proof of proposition, continued

- Equalizer condition: let $\{(U_i, T_i, \delta_i)\} \to (U, T, \delta)$ be a cover.
- $g_i^{-1}\mathcal{F}_{(U,T,\delta)}(T_i) = \mathcal{F}_{(U,T,\delta)}(T_i)$, so $g_{\mathcal{F}}^*$ provides an isomorphism $\mathcal{F}_{(U_i,T_i,\delta_i)}(T_i) \simeq \mathcal{F}_{(U,T,\delta)}(T_i)$.
- Now use equalizer condition for Zariski sheaf $\mathcal{F}_{(U,T,\delta)}$.

Crystalline site: structure sheaf

Proposition

The data of a sheaf $\mathcal{F}$ on $\mathrm{Cris}(X/W_n)$ is equivalent to the data of a Zariski sheaf $\mathcal{F}_{(U,T)}$ on T for each object (U,T), with maps $g_{\mathcal{F}}^{*} : g^{-1}\mathcal{F}_{(U',T')} \to \mathcal{F}_{(U,T)}$ for each morphism $(U,T) \to (U',T')$, satisfying:

- *Transitivity for $(U,T) \to (U',T') \to (U'',T'')$.*
- *If $T \to T'$ is an open immersion and $U = U' \times_{T'} T$, then $g_{\mathcal{F}}^{*}$ is an isomorphism.*

Hence, we can define the structure sheaf $\mathcal{O}_{X/W_n}$ by the rule $(U, T, \delta) \mapsto \mathcal{O}_T$.

Definition

A sheaf of $\mathcal{O}_{X/W_n}$-modules $\mathcal{F}$ is called a crystal if $g_{\mathcal{F}}^{*}$ induces an isomorphism for **every** morphism g.

Crystalline cohomology: take one

- The global sections of a sheaf $\mathcal{F}$ on $\mathrm{Cris}(X/W_n)$ are systems of compatible sections $s_{(U,T,\delta)} \in \mathcal{F}(U,T,\delta)$.
- $H^i(X/W_n)$ is by definition $R^i\Gamma(\mathcal{O}_{X/W_n})$.
- We also set $H^i(X/W) = \varprojlim H^i(X/W_n)$.
- Comparison isomorphism: if Z/W_n is a smooth lift of X, then

$$H^i_{\mathrm{dR}}(Z/W_n) \cong H^i(X/W_n).$$

The relative crystalline site

- Let $(S, \mathcal{I}, \gamma)$ be a PD-scheme with associated closed immersion $S_0 \hookrightarrow S$. We assume $\mathcal{O}_S$ is killed by p^n, so that all PD-thickenings are homeomorphisms.
- We say γ extends to X/S if the diagram

$$\begin{array}{ccc} X \times_S S_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S \end{array}$$

  can be made into a PD-morphism.
- Example: X is an S_0-scheme considered over S. Or, $S = W_n$ and X is any S-scheme.

The relative crystalline site: objects

- Objects are S-morphisms $U \hookrightarrow T$, equipped with PD-structure δ.
- U is still a Zariski open of X. PD-compatibility condition is now that the following is a PD-morphism:

$$\begin{array}{ccc} U \times_S S_0 & \longrightarrow & T \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S \end{array}$$

- Example: $X \to X$ with trivial PD-structure.
- Morphisms and covers are as before.

Digression: divided power envelopes

Given a PD-scheme $(S, \mathcal{I}, \gamma)$ and a closed immersion $X \hookrightarrow Y$ of S-schemes, there is a universal PD-scheme $D_{X,\gamma}(Y)$ such that

$$\begin{array}{ccc} D_{X,\gamma}(Y)_0 & \longrightarrow & D_{X,\gamma}(Y) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

commutes and

$$\begin{array}{ccc} D_{X,\gamma}(Y)_0 \times_S S_0 & \longrightarrow & D_{X,\gamma}(Y) \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S \end{array}$$

is a PD-morphism.

A rigidity theorem

Theorem

Suppose given a Cartesian PD-square:

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S \end{array}$$

There is a natural isomorphism

$$H^i((X/S)_{\mathrm{cris}}, \mathcal{O}_{X/S}) \xrightarrow{\sim} H^i((X_0/S)_{\mathrm{cris}}, \mathcal{O}_{X_0/S}).$$

Passage to topos

- Given a morphism $X \to Y$, it is not true that thickenings of Y pull back to X. E.g. $X \to Y$ could be a nontrivial family.
- To get functoriality, consider the topos $(X/S)_{\mathrm{cris}}$, the category of sheaves on the site $\mathrm{Cris}(X/S)$.
- Grothendieck philoshopy: an object $T \in \mathrm{Cris}(X/S)$ gives rise to a sheaf $\widetilde{T} := \mathrm{Hom}(\cdot, T)$.
- Given $\mathcal{F}$, there is a canonical identification:

$$\mathrm{Hom}(\widetilde{T}, \mathcal{F}) = \mathcal{F}(T).$$

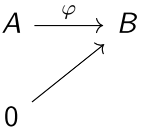
- This embeds $\mathrm{Cris}(X/S)$ into $(X/S)_{\mathrm{cris}}$.

Morphisms of topoi

Definition

A morphism of topoi $f : \mathcal{T}' \to \mathcal{T}$ is a functor $f_* : \mathcal{T}' \to \mathcal{T}$ which has a left adjoint $f^* : \mathcal{T} \to \mathcal{T}'$ commuting with finite inverse limits.

- Intuition: $f^*\mathcal{F}$ and $\mathcal{F}$ are supposed to have the same stalks.
- $\mathrm{Hom}(f^*(\mathcal{F}), \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, f_*(\mathcal{G}))$
- Recall: adjunction automatically means f_* preserves all inverse limits and f^* preserves all direct limits.
- f^* is exact because cokernels are direct limits and $\ker(A \xrightarrow{\varphi} B)$ is the inverse limit of the diagram:



Functoriality of crystalline topos

- Suppose $S' \to S$ is a PD-morphism. We want a morphism $g_{\text{cris}} : (X'/S')_{\text{cris}} \to (X/S)_{\text{cris}}$.

- Writing down a formula is easy; checking the details is hard.

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

Functoriality of crystalline topos

- Given $(U, T, \delta) \in \text{Cris}(X/S)$, $g^*(\widetilde{T})$ is the sheaf on $\text{Cris}(X'/S')$ whose value on (U', T', δ') is:
  - the empty set if $g(U') \not\subset U$;
  - otherwise, the set of PD-morphisms $h : T \to T'$ making the diagram below commute.

$$\begin{array}{ccccc} U' & \longrightarrow & T' & \longrightarrow & S' \\ \downarrow g & & \downarrow h & & \downarrow f \\ U & \longrightarrow & T & \longrightarrow & S \end{array}$$

- Then by definition,

$$g_*(\mathcal{F})(T) = \text{Hom}(\widetilde{T}, g_*\mathcal{F}) = \text{Hom}(g^*(\widetilde{T}), \mathcal{F})$$

Leray spectral sequence

Proposition

Suppose $g : \mathcal{T}' \to \mathcal{T}$ is a morphism of topoi. If E' is an abelian sheaf in $\mathcal{T}'$, there is a Leray spectral sequence:

$$E_2^{pq} = H^p(\mathcal{T}, R^q g_* E') \implies H^i(\mathcal{T}', E').$$

- Alternatively, in derived language, $R\Gamma_{\mathcal{T}'} \simeq R\Gamma_{\mathcal{T}} \circ Rg_*$.

Proof.

Because g^* preserves finite inverse limits, $g^*(e) = e'$, where $e \in \mathcal{T}$ and $e' \in \mathcal{T}'$ are the final objects. So,

$$\Gamma(\mathcal{T}', E') = \mathrm{Hom}_{\mathcal{T}'}(e', E') = \mathrm{Hom}_{\mathcal{T}}(e, g_* E') = \Gamma(\mathcal{T}, g_* E').$$

Also, g_* takes injectives to injectives by general nonsense. So we use the composite functor spectral sequence. □

"Proof" of rigidity theorem

Theorem

Suppose given a Cartesian PD-square:

$$\begin{array}{ccc} X_0 & \xrightarrow{i} & X \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S \end{array}$$

There is a natural isomorphism

$$H^i((X/S)_{\text{cris}}, \mathcal{O}_{X/S}) \xrightarrow{\sim} H^i((X_0/S)_{\text{cris}}, \mathcal{O}_{X_0/S}).$$

- STS: the functor $i_{\text{cris},*}$ is exact and takes $\mathcal{O}_{X_0/S}$ to $\mathcal{O}_{X/S}$.

Comparison isomorphism

Theorem

Suppose given a Cartesian PD-square where X/S is smooth*:*

$$\begin{array}{ccc} X_0 & \xrightarrow{i} & X \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S \end{array}$$

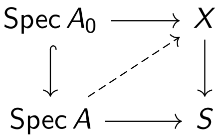
There are natural isomorphisms:

$$H^i((X/S)_{\mathrm{cris}}, \mathcal{O}_{X/S}) \xrightarrow{\sim} \mathbb{H}^i(X_{\mathrm{Zar}}, \Omega^\bullet_{X/S})$$

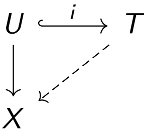
$$H^i((X/S)_{\mathrm{cris}}, \mathcal{O}_{X/S}) \xrightarrow{\sim} H^i((X_0/S)_{\mathrm{cris}}, \mathcal{O}_{X_0/S}).$$

Proof idea 1: the role of smoothness

- Recall that smoothness of X/S means square-zero (hence, nilpotent) affine thickenings lift:



- So, after shrinking, one can always factor:



- X is not a final object, but it is a "covering object". So we can hope to use Cech techniques.

Proof idea 2: passage to Zariski topos

- There is a morphism of topoi $u_{X/S} : (X/S)_{\text{cris}} \to X_{\text{Zar}}$ satisfying:

$$\Gamma(X_{\text{Zar}}, u_{X/S,*}(\mathcal{F})) = \Gamma((X/S)_{\text{cris}}, \mathcal{F})$$

- The Leray spectral sequence then shows

$$H^*((X/S)_{\text{cris}}, \mathcal{O}_{X/S}) = \mathbb{H}^*(X_{\text{Zar}}, Ru_*\mathcal{O}_{X/S}).$$

- One can now use something called the "Cech-Alexander complex".

Crystals and connections

Theorem

Suppose given a Cartesian PD-square where X/S is smooth*:*

$$\begin{array}{ccc} X_0 & \xrightarrow{i} & X \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S \end{array}$$

Then there is an equivalence of categories between

$$\{\textit{crystals on } \operatorname{Cris}(X_0/S)\} \textit{ and}$$

$$\{\textit{QCoh sheaves on } X \textit{ w/ quasi-nilpotent integrable connection } \nabla\}$$

Recall: a sheaf of $\mathcal{O}_{X/W_n}$-modules $\mathcal{F}$ is called a crystal if $g^*_{\mathcal{F}}$ induces an isomorphism for **every** morphism g.

Crystals and connections: sketch

- Suppose given a crystal $\mathcal{F}$. The corresponding sheaf on X will be $\mathcal{F}_X$, which is an $\mathcal{O}_X$-module.
- How to make the connection ∇?
- Consider $\mathcal{P} \subset X \times_S X$, the closed subscheme defined by $\mathcal{I}^2$. This is a thickened neighborhood of $\Delta(X)$.

$$\begin{array}{ccccc} & & & & X \\ & & & \nearrow^{\pi_1} & \\ X & \xrightarrow{\Delta} & \mathcal{P} & & \\ & & & \searrow_{\pi_2} & \\ & & & & X \end{array}$$

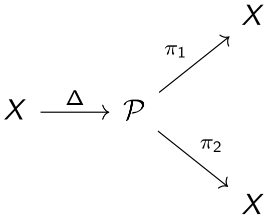
- If this were a PD-thickening, crystal property would give an isomorphism $\epsilon : \pi_1^* \mathcal{F} \xrightarrow{\sim} \pi_2^* \mathcal{F}$

Crystals and connections: sketch

- We can leverage $\epsilon : \pi_1^*\mathcal{F} \xrightarrow{\sim} \pi_2^*\mathcal{F}$ into a connection:

$$\nabla(f) = \epsilon(1 \otimes f) - f \otimes 1 \in \mathcal{F} \otimes \Omega^1_{X/S}$$

- Compatibility with Δ^* means we land in Ω^1.
- Integrability ($\nabla \circ \nabla = 0$) will follow from cocycle condition for $X \times_S X \times_S X$.



Crystals and connections: sketch

- Conversely: given a sheaf $\mathcal{F}_X$ of $\mathcal{O}_X$-modules, how do we make a crystal for X_0/S?
- Because X is smooth, after shrinking we can factor:

$$\begin{array}{ccccc} U & \hookrightarrow & T & \xrightarrow{g} & X \\ \downarrow & & & \searrow & \downarrow \\ X_0 & & & & S \end{array}$$

- We define $\mathcal{F}_T := g^*\mathcal{F}_X$.
- What if we used a different morphism $g' : T \to X$?

Crystals and connections: sketch

$$T \xrightarrow{g \times g'} X \times_S X \quad \begin{array}{l} \xrightarrow{\pi_1} X \\ \xrightarrow{\pi_2} X \end{array}$$

- After passing to a suitable PD-neighborhood, ∇ will give a canonical identification of $g^*\mathcal{F}$ and $(g')^*\mathcal{F}$.
- This lets us construct $\mathcal{F}$.
- Cocycle condition will come from integrability.

Crystals and connections

Theorem

Suppose given a Cartesian PD-square where X/S is smooth*:*

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