Prisms and distinguished elements

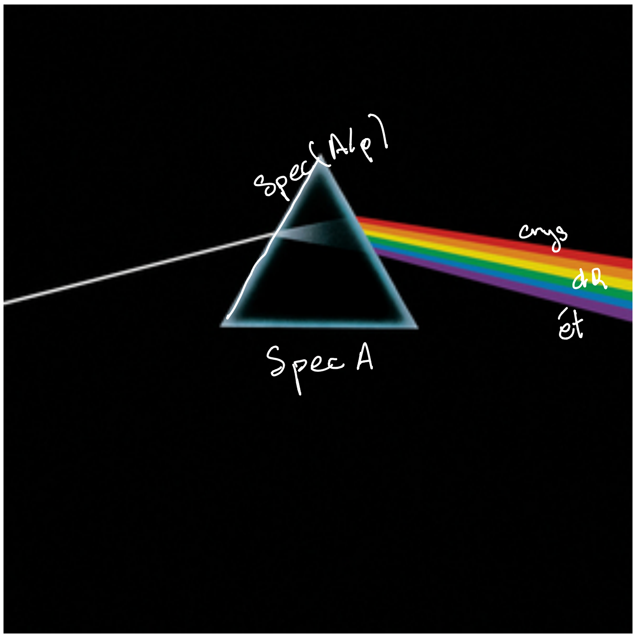


Fig 1: Pink Floyd's original discovery of prismatic cohomology.

<u>Recall</u> Let $X/\mathbb{F}_p$ be sm proper scheme. Then $H^i_{dR}(X/\mathbb{F}_p)$ has a "canonical deformation" to $\mathbb{Z}_p$, called crystalline coh. $H^i_{crys}(X/\mathbb{Z}_p)$, i.e.

- $H^i_{dR}(X/\mathbb{F}_p)$: $\mathbb{F}_p$-module
- $H^i_{crys}(X/\mathbb{Z}_p)$: $\mathbb{Z}_p$-module
- $R\Gamma_{crys}(X/\mathbb{Z}_p) \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p \cong R\Gamma(X, \Omega^*_{X/\mathbb{F}_p})$

cohomology is $H^i_{crys}(X/\mathbb{Z}_p)$

cohomology is $H^i_{dR}(X/\mathbb{F}_p)$

Deformations of cohomology thy's are important!

Prismatic cohomology gives a gen'l framework for making these deformations:

if (A, I) is a <u>(bounded)</u> <u>prism</u> and

ring   ideal of A

$X/(A/I)$ is nice scheme, then $H^i_{\Delta}(X/A)$ is a "canonical deformation" of $H^i_{dR}(X/(A/I))$

If $(A, I) = (\mathbb{Z}_p, (p))$, recover crystalline coh.

<u>Recall</u>

(A, I) : divided-powers ring   prism

R : A/I - algebra

$\mathrm{Cris}(R/A)$ consisted of pd-thickenings

$(R/A)_{\Delta}$ "prismatic thickenings"

$$\begin{tikzcd} B \arrow[r] & B/J = R \\ A \arrow[u] \arrow[r] & A/I \arrow[u] \end{tikzcd}$$

s.t. $(A, I) \longrightarrow (B, J)$ is a pd-morphism.

morphism of prisms

<u>Prisms</u>

<u>Recall</u> A δ-ring A is a ring equipped w/ a set map $\delta : A \to A$ satisfying conditions which

ensure

$$\phi : A \longrightarrow A$$

$$\phi(x) = x^p + p\delta(x)$$

is a ring map lifting the Frobenius on A/p.

A <u>δ-pair</u> (A, I) consists of a δ-ring A and an ideal $I \subseteq A$.

$$\varphi : (A, I) \longrightarrow (B, J)$$

is a δ-ring map w/ $\varphi(I) \subseteq J$.

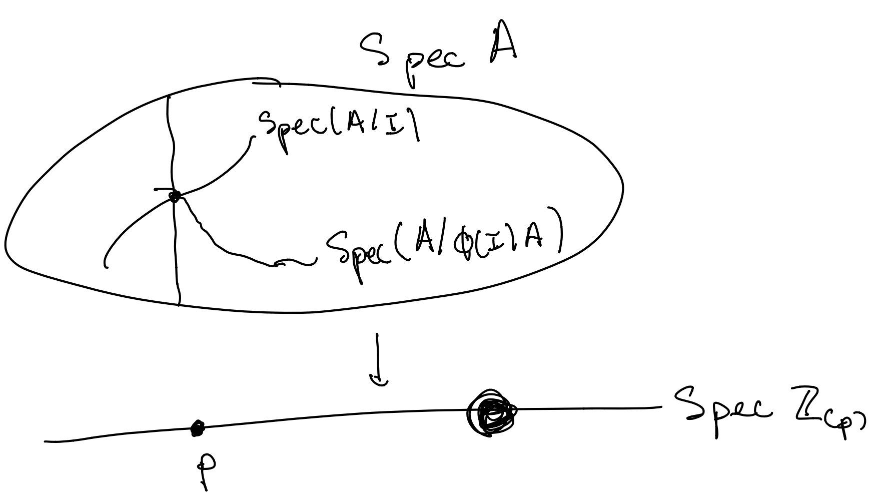
<u>Defn</u>

(1) A δ-pair (A, I) is a <u>prism</u> if

- I defines a Cartier divisor on $\operatorname{Spec}(A)$

i.e. I is locally a principal ideal, generated by a non-zerodivisor

- A is derived (p, I)-complete
- $p \in I + \phi(I)A$.



(2) A prism (A, I) is

(i) <u>perfect</u> if A is a perfect δ-ring, i.e. $\phi : A \to A$ is an iso

(ii) <u>bounded</u> if A/I has bdd p-torsion, i.e. $A/I[p^\infty] = A/I[p^n]$

(iii) <u>crystalline</u> if $I = (p)$.

(3) A map $(A, I) \to (B, J)$ of prisms is (faithfully) flat if $A \to B$ is (p, I)-completely (faithfully) flat, i.e. if $B \otimes^L_A A/(p, I)$ has coh. only in deg 0, given by a flat $A/(p,I)$-module.

<u>Ex.</u> $(\mathbb{Z}_p, (p))$ is a prism

<u>Lem</u> If (A, I) is a bdd prism, then the derived (p, I)-completion of A is the same as the classical (p, I)-completion. (I.e. A is classically (p,I)-complete.)

Same is true for a flat A-module.

<u>Distinguished elements</u>

<u>Recall</u> A: ring, $\mathrm{Rad}(A) = \bigcap_{\substack{m \subseteq A \\ \text{max'l}}} m$ Jacobson radical.

All the rings we deal with will be p-local, i.e. $p \in \mathrm{Rad}(A)$. This is true whenever A is p-complete.

<u>Key fact</u>: if $I \subseteq \mathrm{Rad}(A)$, then $a \in A^\times \iff a \bmod I \in (A/I)^\times$.

<u>Recall</u>: if A is a δ-ring and $Z \subseteq \mathrm{Spec}(A/p)$ is a closed set then the localization A_Z along Z has a unique δ-structure.

<u>Defn</u> A: δ-ring.

$d \in A$ is <u>distinguished</u> if $\delta(d) \in A^\times$.

<u>Obs</u> distinguished elts are preserved by δ-ring maps.

<u>Obs</u> if A is p-local, then $\phi(d)$ dist $\iff$ d dist.

<u>Pf.</u>
$$\begin{aligned}
\delta(d) \in A^\times &\iff \delta(d) \bmod p \in (A/p)^\times \\
&\iff \delta(d)^p \bmod p \in (A/p)^\times \qquad [\phi \text{ and } \delta \text{ commute}] \\
&\iff \delta(\phi(d)) \bmod p \in (A/p)^\times \\
&\iff \delta(\phi(d)) \in A^\times.
\end{aligned}$$

<u>Ex.</u> $A = \mathbb{Z}_{(p)}$, $d = p$ $\quad \delta(p) = 1 - p^{p-1} \in \mathbb{Z}_{(p)}^\times$

In particular, p is distinguished in any p-local δ-ring.

<u>Ex.</u> (The universal dist. elt)

$A = \mathbb{Z}_{(p)}\{d, \delta(d)^{-1}\}$ $=$ $\mathbb{Z}_{(p)}\{d\}$ localized at $\{\delta(d), \phi(\delta(d)), \phi^2(\delta(d)), \dots\}$ is initial among p-local δ-rings w/ a dist. elt.

Ex. Let A: perfect $\mathbb{F}_p$-alg. Recall that $W(A)$ is a p-complete δ-ring via $\phi\left(\sum_{i\geq 0}[a_i]p^i\right) = \sum_{i\geq 0}[a_i^p]p^i$.

Claim:

"$f \in W(A)$ f dist. $\Leftrightarrow \frac{\partial f}{\partial p}$ unit"

$$d = \sum [a_i] p^i \text{ is dist } \Longleftrightarrow a_1 \in A^\times.$$

Pf. $$\delta(d) = \frac{1}{p}\left(\sum [a_i^p] p^i - \left(\sum [a_i] p^i\right)^p\right)$$

$$\equiv [a_1] \quad \text{mod } p$$

so $\delta(d) \in W(A)^\times \Longleftrightarrow a_1 \in (W(A)/p)^\times = A^\times$.

Rmk If A is a perfect p-complete δ-ring then $A \cong W(A/p)$.

Lem Let A: δ-ring, $f \in A$ dist, $u \in A^\times$.

If $f, p \in \mathrm{Rad}(A)$ then uf dist.

Pf.

$$\underset{\substack{\cap \\ A^\times}}{\delta(uf)} = \underset{\substack{\cap \\ A^\times}}{u^p\delta(f)} + \underset{\substack{\cap \\ \mathrm{Rad}(A)}}{f^p\delta(u)} + \underset{\substack{\cap \\ \mathrm{Rad}(A)}}{p\delta(u)\delta(f)}$$

$A^\times \Leftarrow A^\times$ $\square$

Lem Let A: δ-ring, $g \in A$ dist.

Suppose $g = fh$ with $f, p \in \mathrm{Rad}(A)$.

Then f dist and $h \in A^\times$.

"dist elts are irreducible."

Pf. $\delta(g) = f^p\delta(h) + h^p\delta(f) + p\delta(f)\delta(h)$.

$\underset{A^\times}{\rotatebox{90}{$\in$}}$ $\underset{\mathrm{Rad}(A)}{\rotatebox{90}{$\in$}}$ $\Rightarrow$ $\underset{A^\times}{\rotatebox{90}{$\in$}}$ $\underset{\mathrm{Rad}(A)}{\rotatebox{90}{$\in$}}$

$\Rightarrow h \in A^\times$

$\delta(f) \in A^\times$. ☐

Lem Let A : δ-ring, $f, p \in \mathrm{Rad}(A)$. TFAE:

(a) f is dist

(b) $p \in (f^p, \phi(f))$

(c) $p \in (f, \phi(f))$

Pf.

(a) $\Rightarrow$ (b): $\phi(f) = f^p + p\delta(f)$

(b) $\Rightarrow$ (c): ✓

(c) $\Rightarrow$ (a): Suppose $p = af + b\phi(f)$.

WTS $\delta(f) \in A^\times$

$\Updownarrow$

$\delta(f) \bmod (p,f) \in (A/(p,f))^\times$

$\Updownarrow$

$A/(p,f,\delta(f)) = 0$.

Suppose $(p,f,\delta(f)) \neq (1)$. Let $B = A_{V(p,f,\delta(f))}$

so that $p, f, \delta(f) \in \mathrm{Rad}(B)$.

$V(p,f,\delta(f)) = \{\mathfrak{p} \text{ prime} : \mathfrak{p} \supseteq (p,f,\delta(f))\}$.

$p = af + b\phi(f)$

$= af + bf^p + bp\delta(f)$

$\Updownarrow$

$\underbrace{p}_{\text{dist}}\underbrace{(1-b\delta(f))}_{\in B^\times} = f(a+bf^{p-1})$

$\underbrace{\qquad}_{\text{dist}}$ $\Rightarrow f$ dist by lemma.

$\Rightarrow \delta(f) \in B^\times$ contradicting $\delta(f) \in \mathrm{Rad}(B)$. $\square$

<u>Cor</u> Let A: δ-ring, $I \subseteq A$ locally principal, $(p, I) \subseteq \mathrm{Rad}(A)$.

TFAE:

(a) $p \in (I^p, \phi(I))$

(b) $p \in (I, \phi(I))$

(c) "I is pro-Zariski locally gen'd by a dist elt"

That is, $\exists$ faithfully flat δ-ring map $A \to A'$ s.t.

A' = finite product of localizations of A

and $IA' = (f)$ for a dist f with $(p, f) \subseteq \mathrm{Rad}(A')$.

<u>Back to prisms</u>

<u>New ex</u>: $(A, I) = (\mathbb{Z}_p[[u]], (u-p))$ is a prism. $\quad \phi: u \mapsto u^p$

Want $p \in (u-p, u^p - p)$

$\Updownarrow$

$u - p$ is dist. $\quad \delta(u-p) = \frac{1}{p}(u^p - p - (u-p)^p)$

const. term is $1 - p^{p-1} \in \mathbb{Z}_p^\times$.

Lem: Let (A, I) be a prism. Then

(a) $\phi(I)A$ is principal w/ a dist generator.

(b) If (A, I) is perfect, then I is also principal w/ a dist generator.

Prop (Rigidity of maps)

(1) Let $(A, I) \longrightarrow (B, J)$ be a map of prisms.
Then $I \otimes_A B \longrightarrow J$ is an iso, and in particular, $IB = J$.

(2) Conversely, if B is a derived (p, IB)-complete δ-A-algebra for a prism (A, I) then (B, IB) is a prism iff $B[I] = 0$.

Pf of (1) in case I, J are principal:

$I = (f)$ f dist

$J = (g)$ g dist

$IB \subseteq J$

$f = gh$ $h \in B$

Then use irreducibility to get $h \in B^\times$.