The Infinitesimal Site & Algebraic de Rham cohomology

We'd like to use tools of Alg Top to study schemes.

$X/\mathbb{C}$ a smooth variety X^{an} is a complex mfd

$$X/\mathbb{C} \longrightarrow H^*(X^{an}; \mathbb{C})$$

is a reasonable coh. thy for varieties $/\mathbb{C}$.

Can we construct this algebraically.

<u>Yes</u>! Via a version of de Rham's Thm.

Sheaves on X^{an} — hol. diff forms

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d] & \Omega^*_{X^{an}} \\ \mathbb{C} \to \mathcal{O} \to \mathcal{O} \cdots & \end{tikzcd}$$

quasi-iso on stalks by Poincaré Lemma.

$\Updownarrow$

quasi is of sheaves.

We'd like to compute $H^*(X^{an}; \mathbb{C})$ via $\Omega^*_{X^{an}}$.

Hyper Cohomology

Let A an abelian cat w/ enough injectives & $F: A \to B$ a left exact functor.

$$C^\bullet = C^0 \to C^1 \to C^2 \to \cdots$$

There is a hypercohomology of $C^\bullet$ wrt F

$$\underline{F}^i(C)$$

To compute it

(1) Choose a quasi-iso

$C^\bullet \xrightarrow{f} I^\bullet$ where I is a complex of injectives

(2) $\underline{\underline{F}}^i(C^\bullet) \xrightarrow{f_*} \underline{\underline{F}}^i(I^\bullet) = H^i(F(I^\bullet))$

This is • well-defined
• inverts quasi-isomorphisms

If $C^\bullet = C^0 \to 0 \to 0 \cdots$

$\underline{\underline{F}}^i(C^0)$ is the 'usual' derived functor.

<u>Rmk</u> actually $C^\bullet \to A^\bullet \leftarrow A^i$ are F-acyclic.
to compute $\underline{F}^i$

Let $\underline{H}^*$ be $\underline{\Gamma}^*$ ← global sections

Then $H^*(X^{an};\mathbb{C}) = \underline{H}^*(X^{an};\mathbb{C}) = \underline{H}^*(X^{an};\Omega^*_{X^{an}})$

If $X/\mathbb{C}$ is sm/proper.

GAGA $\underline{H}^*(X^{an};\Omega^*_{X^{an}}) = \underline{H}^*(X;\Omega^*_X)$

<u>Rmk</u> proper not necessary.

<u>defn</u> deRham coh of $X/S = \underline{H}^*(X;\Omega^*_{X/S})$

$$\| $$

$$H_{dR}(X/S)$$

If S is characteristic 0, X is smooth, this behaves well

In char p $\Omega^*_{\mathbb{A}^1/k} = k[x] \xrightarrow{d} k[x]\,dx$

acyclic in Zariski topology

$H^*_{dR}(\mathbb{A}^1_k)$ not f.d.!

$f \in k[x]$, $df^p = pf^{p-1}df = 0$

so H^*_{dR} is bad for affine stuff.

In char p, H_{dR} is like $H^*(\ ;\mathbb{Z}/p)$

we might want something like $H^*(\ ;\mathbb{Z}_p)$

p-torsion info

Another problem w/ H^1_{dR} is it's too closely tied to smoothness.

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for $l \neq p$ l-adic coh. captures l-torsion phenomena,

for $l = p$, l-adic coh is bad

$\mathbb{A}^1_k$ has p-fold étale covers. ...

What do we want?

- A Classical Weil coh Thy.

To solve this Grothendieck introduced

- inf infitesmal site char 0

$\rightsquigarrow$

- crystalline site arbitrary char.

Given X/k smooth proper

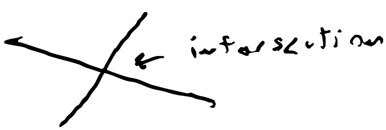
$CH^i(X) =$ (rational) equiv. classes of alg cycles on X of codim i.

formal fin com codim i embedding $Y \to X$

codim $i-1$ $Y' \to X$

f rational fn on Y', $\operatorname{div} f = 0$ in $CH^i(X)$.

- $CH^*(X)$ is a comm. ring



- It is covariant & contravariant

f_*, f^*

$X \xrightarrow{f} Y$.

Weil coh Thy $/F$ ← char 0 field (but let's ignore that)

k ← any char

is the following data

(D1) A functor $H^*: \mathrm{SmProj}^{op}_{/k} \to \mathrm{GrCAlg}_{/F}$ (connected)

$ab = (-1)^{|a||b|} ba$

The cohomology functor

(D2) A natural transformation

$$\gamma: CH^i(X) \to H^{2i}(X)$$

The cycle class map

(D3) A collection of isos. $\int_X : H^{2d}(X) \to F$

trace maps $\quad d = \dim X$

(Stacks project ----)

Satisfying ------

$d = \dim X$

(A) Poincaré Duality

- $H^i(X)$ is f.d, nonzero only if $i \in [0, 2d]$
- $H^i(X) \otimes H^{2d-i}(X) \xrightarrow{\cup} H^{2d}(X) \xrightarrow{\int_X} F$

is a perfect pairing.

$$H^i(X) \xrightarrow{\sim} (H^{2d-i}(X))^\vee$$

(B) Künneth Formula

$$H^*(X) \otimes H^*(Y) \xrightarrow[pr_1^* \otimes pr_2^*]{\sim} H^*(X \times Y)$$

Given (A), we get push forwards $\quad d, e = \dim X, Y$

$X \xrightarrow{f} Y \qquad H^{2d-i}(X) \xrightarrow{f_*} H^{2e-i}(Y)$

(C) compatibility w/ γ.

- $\gamma(f_*) = f_*(\gamma)$

- $\int_{\operatorname{Spec} k} \gamma([\operatorname{Spec} k]) = 1$

These are nice eg:
- A Lefschetz fixed pt formula
- Chern classes
- Prove some Weil conj. (char p)

Examples

ℓ-adic coh. $\quad F = \mathbb{Q}_\ell$

Alg deRham coh $\quad F = k = \text{char } 0 \quad (\text{char } p)?$

Betti coh. $\quad H^\bullet(X^{an}) \quad k = \mathbb{C}$

crystalline coh $\quad F = \mathrm{Frac}(W(k))$

$\parallel$ $\quad k = \mathbb{F}_p$

infinitesmal coh in char 0. $\quad W(k) = \mathbb{Z}_p$

A site theoretic construction of de Rham coh.

de Rham coh. uses

infinitesimal def.

$$X \xrightarrow{\Delta} X \times X$$

$$Z(I) = X, \quad I/I^2 = \Omega^1_X$$

$Z(I^2)$ is a thickening of X

If X/k is $\operatorname{Spec} R$,

$$\operatorname{Hom}(\Omega_{X/k}, M)$$
$$\parallel$$
$$\text{derivations } R \to M$$
$$\parallel$$

$$\begin{tikzcd}
\operatorname{Spec} R \arrow[r] \arrow[d, dashed, bend left] & \operatorname{Spec} k \\
\operatorname{Spec}(R \oplus M) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

thickening of Spec R.

Infinitesimal Site

Let X/S a scheme, S a base, we define X_{inf}

Need to supply

- a category
- a notion of cover

<u>Defn</u> The objects of X_{inf} are

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d] & X \\ T & \end{tikzcd}$$

open subset

← infinitesmal thickening

=

closed inclusion defined by a nilpotent ideal in T.

Morphisms:

$$\begin{tikzcd} U' \arrow[r, hook] \arrow[d] & U \arrow[r, hook] \arrow[d] & X \\ T' \arrow[r] & T & \end{tikzcd}$$

Covers: generated by

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U \arrow[r] \arrow[d] & X \\ T_i \arrow[r] & T & \end{tikzcd}$$

T_i are a Zariski cover of T.

What is a sheaf on X_{inf}?

$$B_i \xrightarrow[\text{cover}]{} A$$

$$\Rightarrow \mathcal{F}(A) \to \prod_i \mathcal{F}(B_i) \rightrightarrows \prod_{ij} \mathcal{F}(B_i \times_A B_j)$$

given
$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \\ T \end{tikzcd}$$

consider $\mathcal{F}(U_i \to T_i)$ where
$$\begin{tikzcd} U_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U \arrow[d] \\ T_i \arrow[r] & T \end{tikzcd}$$
open

This is a Zariski sheaf on T.

called $\mathcal{F}_{U \to T}$.

Given
$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ T' \arrow[r, "g"] & T \end{tikzcd}$$

get $\hat{\mathcal{F}}_{U\to T} \to g_* \hat{\mathcal{F}}_{U'\to T'}$

adjoint to

$$\varphi_g : g^{-1} \hat{\mathcal{F}}_{U\to T} \to \hat{\mathcal{F}}_{U'\to T'}$$

$*$ natural, and if $T' \xrightarrow{g} T$ is an open inclusion, φ_g is an iso.

<u>Prop</u> The data $\hat{\mathcal{F}}_{U\to T}$, φ_g, satisfying

$*$ is the same as a sheaf on X_{inf}

Ex: $\begin{tikzcd} U \arrow[r] \arrow[d] & X \\ T \end{tikzcd} \to \Gamma(\mathcal{O}_T)$ is a sheaf

of rings called $\mathcal{O}_{X_{inf}}$.

Thm (Grothendieck)

Suppose X/S is smooth, S char 0.

Then $\underbrace{H^{\bullet}(X_{inf}; \mathcal{O}_{X_{inf}})}_{\mathrm{Ext}(\mathbb{Z}[\bullet], \mathcal{F})} \cong H^{\bullet}_{dR}(X/S)$

$$H^{\bullet}(G; \mathcal{F}) = \mathrm{Ext}(\mathbb{Z}[G], \mathcal{F})$$

$$\Gamma = \mathrm{Hom}_{AbSh}(\mathbb{Z}[\bullet], \mathcal{F}) = \mathrm{Hom}_{QCoh}(\mathcal{O}_X, \mathcal{F})$$

$$\mathrm{Hom}_{AbSh}(\mathbb{Z}[\bullet], \mathcal{F}) = \mathrm{Hom}_{Sh}(\bullet, \mathcal{F})$$

ringed topos

$(X_{inf}, \mathcal{O}_{X_{inf}})$