Alteration

Goal: to understand the de Jong's result on alteration.

Thm (Hironaka, 1964). Let X be a variety over a field of characteristic zero. Then there is a resolution of singularity $Y \to X$.

Thm (de Jong, 1995). Let X be a variety over any field. Then there is an alteration $Y \to X$ s.t Y is non singular. (regular)

1. Hironaka's Resolution

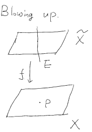
Def. A variety is a integral separated scheme of finite type over some field k.

Def. A morphism of varieties $f: Y \to X$ over k is called a modification iff

1. f is proper, and
2. f is birational

(i.e. there are dense open subschemes $U \subseteq X$ and $V \subseteq Y$ s.t $f|_V : V \to U$ is an isomorphism.)

Eg. Blowing up.



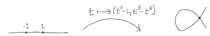
$f|_{\tilde{X} \setminus E} : \tilde{X} \setminus E \to X \setminus P$

is an isomorphism.

Def. A modification $Y \to X$ is called a resolution of singularities iff Y is regular.

Eg. $Y = \operatorname{Spec} k[t]$ $\qquad$ $X = \operatorname{Spec} k[x,y]/(y^2 - x^3 - x^2)$

$$t \longmapsto (t^2-1, t^3-t)$$



Thm. Every curve (1-dim. variety) admits a resolution of singularities.

Def. Given a variety X, let $\operatorname{Reg}(X)$ be the open subscheme of the regular locus of X.

Thm. (Hironaka). Let X be a variety over a field of char. zero. Then there is a resolution of singularities $f: Y \to X$ s.t. f is an isomorphism over $\operatorname{Reg}(X)$.

Qst. Given a variety X, is there a resolution of singularities $Y \to X$?

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2. Normal Crossings.

Def. Let S be a regular scheme. A strict normal crossings divisor (sncd) in S is an effective Cartier divisor $D \subseteq S$ s.t.

1. D is reduced,
2. the irreducible components $\{D_i\}_{i \in I}$ of D are regular, and
3. for any finite subset $J \subseteq I$,

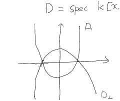
$$D_J = \bigcap_{i \in J} D_i$$

is regular of codimension $\# J$.

(i.e. for all $x \in D_J$,

$$\text{Krull Dim } \mathcal{O}_{D_J, x} = \dim \mathcal{O}_{S,x} - \# J).$$

Eg. Let $S = \mathbb{A}^2$ and
$$D = \operatorname{Spec} k[x,y]/\big((y-x^2+1)(y+x^2-1)\big).$$



Eg. Let $S = \mathbb{A}^2$ and
$$D = \operatorname{Spec} k[x,y]/(y^2-x^3-x^2).$$
Then D is not an sncd, since D is not regular.

However, $\operatorname{Spec} \widehat{\mathcal{O}}_{D,(0,0)} \subseteq \operatorname{Spec} \widehat{\mathcal{O}}_{\mathbb{A}^2,(0,0)}$ is an sncd.

Def. A reduced effective Cartier divisor D in a regular scheme S is a normal crossings divisor (ncd) if for all $x \in D$, there is an étale neighborhood $S' \to S$ s.t.
$$S' \times_S D \subseteq S' \text{ is an sncd.}$$

Rmk. If S is a variety (or more generally an excellent scheme), then $D \subseteq S$ is an ncd iff for all $x \in D$,
$$\operatorname{Spec} \widehat{\mathcal{O}}_{D,x} \subseteq \operatorname{Spec} \widehat{\mathcal{O}}_{S,x}$$
is an ncd.

In particular, $D = \operatorname{Spec} k[x,y]/(y^2-x^3-x^2) \subseteq \mathbb{A}^2$ is an ncd.

Thm. (Hironaka). A variety X over a field of char. 0 has a resolution of singularities $f: Y \to X$ s.t.

1. f is an isomorphism over $\operatorname{Reg}(X)$, and
2. $(Y \setminus f^{-1}(\operatorname{Reg}(X)))_{\mathrm{red}} \subseteq Y$ is an sncd.

3. Alterations

Def. A morphism $f: Y \to X$ of integral noetherian schemes is an alteration iff f is

1. proper,
2. dominant, and
3. generically finite (i.e. $[k(Y): k(X)] < \infty$)

Thm (de Jong) Let X be a variety. Then there is an alteration $f: Y \to X$ and an open immersion $Y \hookrightarrow \overline{Y}$ s.t.

1. $\overline{Y}$ is regular and projective.
2. $\overline{Y} \setminus Y \subseteq \overline{Y}$ is an sncd.

Rmk. The projectivity of $\overline{Y}$ follows from Chow's lemma.

Lem (Chow). Given a variety X, there is a modification $Y \to X$ such that Y is quasi-projective.

Thm (de Jong). Let X be a variety. Let $Z \subseteq X$ be a proper closed subset. Then there is an alteration $f: Y \to X$ and an open immersion $Y \hookrightarrow \overline{Y}$ s.t.

1. $\overline{Y}$ " "
2. $f^{-1}(Z) \cup (\overline{Y} \setminus Y) \subseteq \overline{Y}$ is an sncd.

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4. Regular vs Smooth.

Def. A scheme X is regular, if all local rings $\mathcal{O}_{X,x}$ is regular.

Def. A scheme X over k is smooth over k if $X_{\overline{k}}$ is regular.

Rmk. Smoothness implies regularity. The converse is not true if k is imperfect.

Eg (MacLane). Let $k = \mathbb{F}_p(s,t)$ and $A = k[x,y]/(sx^p+ty^p-1)$.

Then 1. A is a Dedekind domain

2. $\operatorname{Spec} A \otimes_k \bar{k}$ is nonreduced.

$$\left(sx^p+ty^p-1 = (s^{1/p}x+t^{1/p}y-1)^p \text{ in } \bar{k} \right)$$

Rmk. When k is imperfect, then a variety may not have a alteration with a smooth var. Let $X = \operatorname{Spec} A$, and suppose we have an $f: Y \to X$ s.t $\boxed{Y \text{ is smooth}}$.

$$\begin{tikzcd} V_{\bar{k}} \arrow[r, hook] \arrow[d, "\text{finite flat}"'] & Y_{\bar{k}} \arrow[d, "f"] \\ U_{\bar{k}} \arrow[r, hook] & X_{\bar{k}} \end{tikzcd}$$

Then $U_{\bar{k}}$ should be smooth. The contradicts the fact that $X_{\bar{k}}$ is nonreduced.

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5. Very Very Rough Outline of the proof.

- Reduce to the case where X is quasi-projective and k is algebraically closed.
- Use induction on $\dim X$. If $\dim X = 1$, then we have a res. sing.
- By blowing up X, reduce to the case where X has a fibration $X \to B$ s.t all fibers are curves.
  $\dim B = \dim X - 1$
- After an alteration of B, we reduce to the case which $X \to B$ s.t. all fibers only have ordinary nodes as singularities.
- By induction hypothesis, we may assume that B is regular.
- Explicitly resolve the singularities.

$$X \to B$$