Period maps and the Gauss–Manin connection

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Abstract. For a family of smooth projective varieties over a number field, we have a complex period map and a p-adic period map, and they are both governed by the (algebraic) Gauss–Manin connection. After some preliminaries, we introduce these objects and prove some bounds on the dimensions of their images.

Contents

1. Introduction

1.1. Where we are and goals For the whole talk, we fix:

- K a number field.
- S a finite set of places of K (containing the archimedean places).
- $\mathcal{O}$ the ring of S-integers of K.
- $p > 2$ a rational prime and v a place of K over p, such that $K_v/\mathbb{Q}_p$ is unramified and p does not lie below any place in S.

Let Y/K be a smooth projective variety and let $X \to Y$ be a smooth proper family with a smooth proper model $\mathcal{X} \to \mathcal{Y}$ over $\mathcal{O}$. Our goal is to bound $\mathcal{Y}(\mathcal{O})$. The strategy is to see that there are only finitely many possibilities for certain "Hodge theoretic data" and there are only finitely many points that yield a fixed such datum.

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The period map describes how the Hodge theory of a family of varieties varies across its fibers. More specifically, the de Rham cohomologies of these fibers can be identified via the Gauss–Manin connection; but the Hodge structures on this common de Rham cohomology are different, and the period map tells us how it changes.

In our situation over K, we can base change to either $\mathbb{C}$ or K_v (fix some embeddings), and look at either the complex Hodge theory or the p-adic Hodge theory, and their associated period maps. These are compatible in the appropriate sense, so we can prove things about the latter using the former, where more topological machinery is available.

1.2. Review of complex geometry (Reference: Voisin [Voi02].)

Hodge theory Let $X/\mathbb{C}$ be a smooth projective (hence compact Kähler) variety. The *Hodge decomposition* for X is

$$H^i_{dR}(X/\mathbb{C}) \simeq \bigoplus_{p+q=i} H^q(X, \Omega^p_{X/\mathbb{C}}) =: \bigoplus_{p+q=i} H^{p,q}(X/\mathbb{C}).$$

This decomposition satisfies *Hodge symmetry*: $H^{p,q}(X/\mathbb{C}) \simeq \overline{H^{q,p}(X/\mathbb{C})}$. The proof of the Hodge decomposition uses some hard analysis (properties of elliptic operators). What we can recover algebraically is the *Hodge filtration*,

$$F^j H^i_{dR}(X/\mathbb{C}) \coloneqq \bigoplus_{\substack{p+q=i \\ p \geqslant j}} H^{p,q}(X/\mathbb{C})$$

This is because the filtration comes from the degeneration of the Hodge-de-Rham spectral sequence

$$H^q(X, \Omega^p_{X/\mathbb{C}}) \Rightarrow H^{p+q}_{dR}(X/\mathbb{C}).$$

Deligne–Illusie proved this purely algebraically for smooth proper varieties X in characteristic 0 (but their proof goes through characteristic p!). Of course a filtration is the same thing as some flag in $H^\bullet_{dR}$, but not all flags correspond to a Hodge filtration. For example, we also need an analog of Hodge symmetry.

The benefit of using the filtration is that one does not need to work over $\mathbb{C}$—the Hodge filtration exists for the de Rham cohomology even if X is defined over $\mathbb{Q}_p$ or K_v.

Ehresmann's theorem Let $\pi\colon X \to S$ be a smooth proper surjection of $\mathbb{C}$-manifolds.

Theorem 1.1 (Ehresmann)**.** *π is a fiber bundle.*

If S is simply connected then Ehresmann's theorem tells us there is a diffeomorphism $X \simeq S \times X_o$ for any fiber X_o ($o \in S$). In particular, the de Rham cohomologies of the fibers are constant, because they only depend on the underlying topological space. This is the starting point for the Gauss–Manin connection.

The diffeomorphism $X \simeq S \times X_o$ might not be analytic. Since the Hodge filtration depends essentially on the complex structure, it might not be preserved across fibers. This is the starting point for the period map.

2. The Gauss–Manin connection

There is a purely algebraic way to define the Gauss–Manin connection; but we will need to make use of some topology to bound things later. So we also give the more geometric (but topological) definition first.

2.1. Riemann–Hilbert Let X be a connected complex manifold. Recall that a *local system* on X is a locally constant sheaf of $\mathbb{C}$-vector spaces on X.

Monodromy Let H be a local system on X and fix some $x \in X$. Then I claim $\pi_1(X, x)$ acts on the fiber H_x. More generally, any local system on $[0, 1]$ is trivial, so a path from a to b in H induces an isomorphism $H_a \to H_b$. This representation $\pi_1(X, x) \to GL(H_x)$ is called the *monodromy representation*.

Theorem 2.1. *There is an equivalence of categories*

$$\{\text{complex representations of } \pi_1(X, x)\} \simeq \{\text{local systems on } X\}.$$

Sketch. We already constructed the functor in one direction. For the other, note that a π_1-representation defines a locally constant π_1-equivariant sheaf on the universal cover $\widetilde{X}$, which defines a locally constant sheaf on X. □

Connections Let $E \to X$ be a complex vector bundle, or equivalently a locally free $\mathcal{O}_X$-module $\mathcal{E}$ (translation: $\mathcal{E} = \Gamma(-, E)$) A *connection* on E is a $\mathbb{C}$-linear map

$$\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{C}}$$

satisfying the Leibniz rule: $\nabla(as) = a\nabla s + s \otimes da$ for any section $s \in \mathcal{E}$ and function $f \in \mathcal{O}_X$. We can inductively extend ∇ to $\mathcal{E} \otimes \Omega^p_{X/\mathbb{C}}$ by forcing the Leibniz rule:

$$\nabla(s \otimes \alpha) \coloneqq (\nabla s) \wedge \alpha + s \otimes d\alpha.$$

If $\nabla^2\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_{X/\mathbb{C}}$ is zero, we say ∇ is an *integrable connection*.

Theorem 2.2 (Riemann–Hilbert lite)**.** *There is an equivalence of categories*

$$\{\text{vector bundles on } X \text{ with integrable connection}\} \simeq \{\text{local systems on } X\}.$$

Sketch. We give the functors but don't check that they are inverse (not obvious). Given a vector bundle with integrable connection $(\mathcal{E}, \nabla)$, our local system is given by the flat sections, i.e. $H \coloneqq \mathcal{E}^{\nabla=0}$. Given a local system H, we can form a vector bundle $H \otimes_{\mathbb{C}} \mathcal{O}_X$, with connection given by $fs \mapsto s \otimes df$. □

This is a simple version of the Riemann–Hilbert correspondence. The full version gives an equivalence between the derived categories of regular holonomic D-modules ("systems of PDEs") and of constructible sheaves. Integrable connections are the simplest kinds of D-modules and correspond to local systems. See e.g. [HTT08] for more.

2.2. The topological Gauss–Manin connection (Reference: [Voi02].)

Let $\pi\colon X \to S$ be a smooth proper surjective morphism of complex manifolds. By Ehresmann's theorem, the sheaves $R^q\pi_*\underline{\mathbb{C}}_X$ are locally constant. Thus, by Riemann–Hilbert, we obtain a locally free $\mathcal{O}_X$-module

$$\mathcal{H}^q \coloneqq R^q\pi_*\underline{\mathbb{C}}_X \otimes_{\mathbb{C}} \mathcal{O}_S$$

equipped with an integrable connection $\nabla\colon \mathcal{H}^q \to \mathcal{H}^q \otimes_{\mathcal{O}_S} \Omega^1_{S/\mathbb{C}}$. This is the *Gauss–Manin connection*.

To understand what this connection is doing, notice that, by proper base-change, the fiber at $s \in S$ of $\mathcal{H}^q$ is just $H^q(X_s, \underline{\mathbb{C}}_{X_s}) \simeq H^q_{dR}(X_s/\mathbb{C})$. So a section σ of $\mathcal{H}^q$ over some simply connected $\Omega \subset S$ is a (holomorphically varying) family of cohomology classes

$$[\omega_s] \in H^q_{dR}(X_s/\mathbb{C}) : s \in \Omega,$$

and ∇ simply gives us a way to "differentiate" ω_s with respect to s (this is what a connection does and why it is a called a connection).

Moreover, σ is flat (i.e. $\nabla\sigma = 0$) if and only if it is a constant section of $\mathcal{H}^q$, i.e. the ω_b represent the same cohomology class after identifying all the $H^q_{dR}(X_s/\mathbb{C})$ for $s \in \Omega$ via Ehresmann's theorem. That is, if we fix some $o \in \Omega$, then parallel transport along ∇ induces an isomorphism

$$\mathrm{GM}\colon H^q_{dR}(X_o/\mathbb{C}) \xrightarrow{\sim} H^q_{dR}(X_s/\mathbb{C})$$

for any other $s \in \Omega$. The section σ is flat if and only if $\mathrm{GM}(\omega_o) = \omega_b$ for all $s \in S$.

2.3. The algebraic Gauss–Manin connection (Reference: [KO68].)

We have defined the Gauss–Manin connection ∇ and the map GM for maps of complex varieties, but we would like to be able to do this over K directly; this would enable us to obtain compatible connections after base-change to either K_ν or $\mathbb{C}$. In other words, we would like to define an *algebraic* definition (without Ehresmann's theorem; this also lets us get rid of the proper hypothesis).

Let $\pi\colon X \to S$ be a smooth morphism of smooth schemes over a field k. We define a (decreasing) filtration on $\Omega^\bullet_{X/k}$ by

$$F^i\Omega^j_{X/k} \coloneqq \mathrm{im}\left(\pi^*\Omega^i_{S/k} \otimes_{\mathcal{O}_X} \Omega^{j-i}_{X/k} \to \Omega^j_{X/k}\right).$$

That is, "forms in which i coordinates come from S."

The exact sequence

$$0 \to \pi^*\Omega^1_{S/k} \to \Omega^1_{X/k} \to \Omega^1_{X/S} \to 0$$

lets us compute the associated grades as $\mathrm{gr}^i_F\,\Omega^j_{X/k} \simeq \pi^*\Omega^i_{S/k} \otimes_{\mathcal{O}_X} \Omega^{j-i}_{X/S}$. The spectral sequence of a filtered complex to compute $\mathbb{R}\pi_*\Omega^\bullet_{X/k}$ then gives

$$E_1^{p,q} = \Omega^p_{S/k} \otimes_{\mathcal{O}_S} \mathcal{H}^q \Rightarrow R^{p+q}\pi_*\Omega^\bullet_{X/k},$$

where $\mathcal{H}^q = R^q\pi_*\Omega^\bullet_{X/S}$. Consider the E_1 differentials

$$0 \to \mathcal{H}^q \xrightarrow{d_1^{0,q}} \mathcal{H}^q \otimes_{\mathcal{O}_S} \Omega^1_{S/k} \to \mathcal{H}^q \otimes_{\mathcal{O}_S} \Omega^2_{S/k} \to \mathcal{H}^q \otimes_{\mathcal{O}_S} \Omega^3_{S/k} \to \dots$$

Theorem 2.3 ([KO68, Theorem 1]). *The map $\nabla \coloneqq d_1^{0,q}$ is a connection, and all other maps in this sequence are induced from ∇.*

Sketch. The key insight is that F is compatible with the wedge product: $F^p \wedge F^q \subset F^{p+q}$. This descends to a wedge product on the spectral sequence itself:

$$E_r^{p,q} \times E_r^{p',q'} \to E_r^{p+p',q+q'},$$

In other words, $E_r^{\bullet,\bullet}$ is a graded commutative algebra, and moreover the differentials obey the Leibniz rule therein.

Now, when $q = 0$ the result is clear because $d_1^{\bullet,0}$ is just the de Rham differential. The rest follows from the Leibniz rule. ☐

Theorem 2.4 ([KO68, Theorem 2]). *This new connection coincides with the Gauss–Manin connection defined in the previous section over $\mathbb{C}$.*

I omit the proof; it can be obtained either as a consequence of (a stronger form of) Riemann–Hilbert, or of a careful calculation with coordinates (as in [KO68]).

2.4. Residue discs (Reference: [LV20, Section 3.3].)

Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a smooth proper morphism of smooth varieties over $\mathcal{O}$. We have defined a Gauss–Manin connection on $\mathcal{H}^q \coloneqq$ at the generic point of $\mathcal{O}$ (i.e., over K), and by standard arguments this extends to $\mathcal{O}$ after inverting finitely many primes (i.e., enlarging S), so we assume this is the case to begin with: we have

$$\nabla\colon \mathcal{H}^q \to \mathcal{H}^q \otimes_{\mathcal{O}_\mathcal{Y}} \Omega^1_{\mathcal{Y}/\mathcal{O}}.$$

For any point $y \in \mathcal{Y}(\mathcal{O})$, we can look at the fiber $X_{y,\mathbb{C}} \coloneqq \pi^{-1}(y) \times_{\mathcal{O}} \mathbb{C}$. Just as we did in the complex case, we would like to identify $H_{dR}(X_{y,\mathbb{C}})$ for all y "near" some fixed point $o \in \mathcal{Y}(\mathcal{O})$ using the Gauss–Manin connection. In particular, we need to figure out what "near" means. This comes down to some explicit calculations with power series.

Key computation Fix some $o \in \mathcal{Y}(\mathcal{O})$ and let $\overline{o} \in \mathcal{Y}(\mathbb{F}_v)$ be its reduction modulo v. Then by commutative algebra, there is a regular sequence $p, z_1, \dots, z_m \in \mathcal{O}_{\mathcal{Y},\overline{o}}$ generating the maximal ideal, such that

- $(z_1, \dots, z_m) = \ker\left(\mathcal{O}_{\mathcal{Y},\overline{o}} \xrightarrow{o} \mathcal{O}_{(v)}\right)$.
- $\widehat{\mathcal{O}}_{\mathcal{Y},\overline{o}} \simeq \mathcal{O}_v[\![z_1, \dots, z_m]\!]$.
- The image of $\mathcal{O}_{\mathcal{Y},o}$ lies in $\mathcal{O}_{(v)}[\![z_1, \dots, z_m]\!]$.
- $\Omega^1_{\mathcal{Y}/\mathcal{O}}$ is generated by dz_i, where defined.

We should think of the z_i as local coordinates near $\overline{o}$.

Pick a basis over $k(\overline{o})$ for the fiber of $\mathcal{H}^q$ at o and lift it to a basis $v_1, \dots, v_r$ of the stalk (over $\mathcal{O}_{\mathcal{Y},\overline{o}}$). Then, near $\overline{o}$, we can write

$$\nabla v_i = \sum_j v_j \otimes A_{ij}$$

for some $A_{ij} \in \Omega^1_{\mathcal{Y}/\mathcal{O}}$, or in coordinates $A_{ij} = \sum_k a_{ijk} dz_k$ for some $a_{ijk} \in \mathcal{O}_{\mathcal{Y},\overline{o}}$ Then a section $\sum f_i v_i$ is flat if and only if

$$df_i + \sum_j A_{ij} f_j = df_i + \sum_{j,k} a_{ijk} f_j dz_k = 0$$

for each i. We can solve this system formally in the complete rings $\mathcal{O}_v [\![z_1, \dots, z_m]\!]$ and $\mathbb{C} [\![z_1, \dots, z_m]\!]$, since the coefficients a_{ijk} land in $\mathcal{O}_{(v)} [\![z_1, \dots, z_m]\!]$. If the solutions converge in some v-adic (resp. complex) neighborhood Ω_v (resp. $\Omega_{\mathbb{C}}$) of $\overline{o}$, then we obtain a basis of flat sections for $\mathcal{H}$ in that neighborhood, which enables us to define the identifications

$$\begin{aligned} &\mathrm{GM}\colon H_{dR}(X_{o,v}/K_v) \xrightarrow{\sim} H_{dR}(X_{y,v}/K_v) && \text{for all } y \in \Omega_v, \\ &\mathrm{GM}\colon H_{dR}(X_{o,\mathbb{C}}/\mathbb{C}) \xrightarrow{\sim} H_{dR}(X_{y,\mathbb{C}}/\mathbb{C}) && \text{for all } y \in \Omega_{\mathbb{C}}. \end{aligned}$$

Example 2.5. Suppose $m = r = 1$. Then the system of equations becomes

$$df(z) + a(z) f(z) dz = 0$$

and the solutions are spanned by $\exp\left(-\int a(z)dz\right)$. When does this converge? In the complex case it converges whenever $a(z)$ does, so certainly in some small enough neighborhood of o. In the v-adic case it converges whenever the argument of exp lies in the disc $|\bullet|_v < |p|^{\frac{1}{p-1}}$. This is automatically true when $|z|_v < |p|_v^{\frac{p}{p-1}}$. In particular, if $y \equiv o$ modulo v, then this holds (because $p > 2$ and is unramified).

The general case is similar to the example (it is a matrix differential equation). We define the *residue discs*

$$\Omega_v = \{y \in \mathcal{Y}(\mathcal{O}) : y \equiv o \mod v\} \quad \text{and} \quad \Omega_{\mathbb{C}} = \{y \in Y_{\mathbb{C}}(\mathbb{C}) : |y - o| < \epsilon\},$$

where ϵ is chosen small enough to ensure convergence as above. We thus have an identification GM of the v-adic (resp. complex) de Rham cohomologies of the fibers over Ω_v (resp. $\Omega_{\mathbb{C}}$).

Universal covers Let $\widetilde{Y_{\mathbb{C}}(\mathbb{C})} \to Y_{\mathbb{C}}(\mathbb{C})$ be the universal cover of the complex manifold $Y_{\mathbb{C}}(\mathbb{C})$. For $y \in \widetilde{Y_{\mathbb{C}}(\mathbb{C})}$, we continue to write $X_{y,\mathbb{C}}$ for the fiber over the image of y in $Y_{\mathbb{C}}(\mathbb{C})$ (equivalently, pull back $X_{\mathbb{C}}(\mathbb{C})$ along the universal cover).

Fix a point $\widetilde{o} \in \widetilde{Y_{\mathbb{C}}(\mathbb{C})}$ lifting o. Notice that GM extends uniquely identification

$$\mathrm{GM}\colon H^1_{dR}(X_o/\mathbb{C}) = H^q_{dR}(X_{\widetilde{o}}/\mathbb{C}) \xrightarrow{\sim} H^q_{dR}(X_y/\mathbb{C})$$

for all $y \in \widetilde{Y_{\mathbb{C}}(\mathbb{C})}$. This is because GM exists locally and the simply connectedness of $\widetilde{Y_{\mathbb{C}}(\mathbb{C})}$ ensures that there is only one way to extend it globally.

3. Period maps

3.1. Flag varieties Let V be a vector space (over some field k) and let $0 < d_1 < \cdots < d_n < \dim V$ be integers. Then we can construct a space of flags $\mathcal{H}$ whose points correspond to the following families of subspaces

$$\{0 \subset V_1 \subset V_2 \subset \dots V_n \subset V : \dim V_i = d_i\}.$$

Concretely, we can realize $\mathcal{H}$ as a closed subspace of the product of grassmannians

$$\mathrm{Gr}(d_1, V) \times \mathrm{Gr}(d_2, V) \times \cdots \times \mathrm{Gr}(d_n, V).$$

So $\mathcal{H}$ is a projective variety.

3.2. Variations in Hodge structure Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a smooth proper morphism of smooth varieties over $\mathcal{O}$. Fix a point $o \in \mathcal{Y}(\mathcal{O})$ and $L \in \{K_v, \mathbb{C}\}$. We have seen that the Gauss–Manin connection gives identifications

$$\mathrm{GM}\colon H^q_{\mathrm{dR}}(X_{o,L}/L) \xrightarrow{\sim} H^q_{\mathrm{dR}}(X_{y,L}/L)$$

for any y in the residue disc Ω_L. We also have a Hodge filtration on $H^q_{\mathrm{dR}}(X_{y,L}/L)$, which we can transport to o via GM. Thus, we get a map

$$\begin{aligned} \Phi_L\colon \Omega_L &\to \mathcal{H}_L \\ y &\mapsto \mathrm{GM}^{-1}(F^\bullet H^q(X_{y,L}/L)), \end{aligned}$$

which is called the *period map*. Here $\mathcal{H}$ is the space of flags in $H^q_{\mathrm{dR}}(X_{o,K}/K)$ defined by the integers $d_i \coloneqq \dim F^i H^q_{\mathrm{dR}}(X_{o,K}/K)$. The following fact is crucial:

Proposition 3.1. *The period map is analytic. The power series that define it in some local coordinate system have coefficients in* K.

Sketch. Fix some i and notice that there is a bundle $F^i\mathcal{H}^q \subset \mathcal{H}^q$ on Y coming from the Hodge filtration. Taking wedge powers we get a map

$$\mathbb{P}(\Lambda^{d_i} F^i \mathcal{H}^q) \to \mathbb{P}(\Lambda^{d_i}\mathcal{H}^q).$$

The first bundle is trivial, so we get a map $Y \to \mathbb{P}(\Lambda^{d_i}\mathcal{H}^q)$, which actually lands in the grassmannian bundle $\mathrm{Gr}(d_i, \mathcal{H}^q)$ via the Plücker embedding. Everything so far was algebraic and defined over K.

Now, in some residue disc, we can trivialize and send everything to the fiber over o, via GM. The result follows because GM is defined via power series over K. □

We note that the complex period map extends to the universal cover $\widetilde{Y_{\mathbb{C}}(\mathbb{C})}$.

3.3. Bounding the image We continue with $\pi\colon \mathcal{X} \to \mathcal{Y}$, residue discs $\Omega_{\mathbb{C}}$, Ω_v, period maps $\Phi_{\mathbb{C}}$, Φ_v, and flag variety $\mathcal{H}$ as above.

Recall that the fundamental group $\pi_1 \coloneqq \pi_1(Y_{\mathbb{C}}(\mathbb{C}), o)$ acts on $H^q_{\mathrm{dR}}(X_{o,\mathbb{C}}/\mathbb{C})$ by monodromy. We get a representation $\pi_1 \to \mathrm{GL}(H^q_{\mathrm{dR}}(X_{o,\mathbb{C}}/\mathbb{C}))$, let Γ be the Zariski closure of the image. Notice that Γ acts on $\mathcal{H}_{\mathbb{C}}$.

Lemma 3.2 ([LV20, 3.1-3.3])**.** *In the above notation,*

$$\dim\left(\Gamma \cdot \Phi_{\mathbb{C}}(o)\right) \leqslant \dim \overline{\Phi_{\mathbb{C}}(\widetilde{Y_{\mathbb{C}}(\mathbb{C})})} = \dim \overline{\Phi_{\mathbb{C}}(\Omega_{\mathbb{C}})} = \dim \overline{\Phi_v(\Omega_v)}$$

Sketch. The first follows because $\Phi_{\mathbb{C}}$ is $\pi_1(\widetilde{Y_{\mathbb{C}}(\mathbb{C})}, \tilde{o})$-equivariant.

For the middle equality, we start with $\Phi_{\mathbb{C}}^{-1}(\overline{\Phi_{\mathbb{C}}(\Omega_{\mathbb{C}})}) \supset \Omega_{\mathbb{C}}$. The left-hand side is a closed analytic subspace of $\widetilde{Y_{\mathbb{C}}(\mathbb{C})}$ and the right-hand side has the same dimension as $\widetilde{Y_{\mathbb{C}}(\mathbb{C})}$; so the left-hand side must be all of $\widetilde{Y_{\mathbb{C}}(\mathbb{C})}$.

The last equality follows because the formal power series defining $\Phi_{\mathbb{C}}$ and Φ_v coincide: it can be checked that this, together with the fact that $\mathcal{H}$ is projective, implies that the Zariski closures of their images are both obtained by base-change from some closed $Z \subset \mathcal{H}$ defined over K. So they have the same dimension. $\square$

Thus, if $Z \subset \mathcal{H}_v$ has dimension strictly smaller than that of $\Gamma \cdot h_o$, then $\overline{\Phi_v^{-1}(Z)}$ is a Zariski-closed proper analytic subspace of Ω_v. So in the Mordell case (on a curve) it is finite! In the next talk, using p-adic Hodge theory, we will figure out the right Z to deduce the finiteness of $\mathcal{Y}(\mathcal{O})$.

4. p-adic geometry

Recall that in the complex case we could extend the period map to the universal cover. This is what enabled us to use monodromy to lower bound the image. In the p-adic case however, we stuck to the tiny disc Ω_v. Can we do better?

It turns out we can, with the fancy technology of Scholze's diamonds. These are some kind of nonarchimedean analogue of algebraic spaces. We have the following theorem:

Theorem 4.1 ([Han17]). *Let X be a smooth connected rigid analytic space over a p-adic field E and V be a de Rham local system. Then there is a* Hodge–Tate period map

$$\pi_{\mathrm{HT}} \colon \mathcal{T}\mathrm{riv}_{V/X^\diamond} \to \mathcal{F}\ell^\diamond$$

of diamonds over Spd E. *Here $\mathcal{F}\ell$ is a suitably chosen generalized flag variety and $\mathcal{T}\mathrm{riv}_{V/X^\diamond}$ is a pro-étale torsor over $X^\diamond$ (like a covering space).*

One should be careful about what one means by "local system"; here "de Rham" has roughly the same meaning it has in (relative) p-adic Hodge theory. This approach was earlier used to compute the cohomology of perfectoid Shimura varieties [CS24, She16].

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