STAGE: a 90-min crash course on étale cohomology

Matthew Hase-Liu

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1 Introduction

This is an attempt to cover some fundamental aspects of étale cohomology in 90 minutes for STAGE, which is on the Weil conjectures this semester. There are many great references out there, among which I found Tamme's *Introduction to étale cohomology* very friendly as an introduction (at least compared to any of the original articles) up to defining l-adic cohomology. I've listed a couple of other notes and books in the references, which were helpful in learning the material and writing up these notes.[1]

[1] I would also like to thank Prof. Poonen for taking the time to help me prepare my talk and for suggesting lots of very insightful feedback!

Definition 1. A **site** comprises the data of a category $\mathcal{C}$ and a collection of families $\{U_i \to U\}_{i \in I}$ of morphisms in $\mathcal{C}$, denoted $\operatorname{Cov}(\mathcal{C})$ (also called the **Grothendieck (pre-)topology** whose elements are **covering families**), satisfying the following axioms:

(i) (Existence of fiber products) Given $U_i \to U$ in some covering family and any morphism $V \to U$, the fiber product $U_i \times_U V$ exists in $\mathcal{C}$.

(ii) (Stability under base change) Given a covering family $\{U_i \to U\}_{i \in I}$ and any morphism $V \to U$, the collection $\{U_i \times_U V \to V\}_{i \in I}$ is a covering family as well.

(iii) (Local character) Given a covering family $\{U_i \to U\}_{i \in I}$ and for each $i \in I$, another covering family $\{U_{ij} \to U_i\}_{j \in J_i}$, the family of composites $\{U_{ij} \to U\}_{i \in I, j \in J_i}$ is also a covering family.

(iv) (Isomorphisms) If $f : V \to U$ is an isomorphism, then $\{f\}$ is a covering family.

As we shall below, this construction is, indeed, a generalization of a topology.

Example 2. Let X be a topological space and $\mathsf{Op}(X)$ be the category of open sets of X, whose morphisms are given by inclusions. Covering families are then given by (topological) coverings $\{U_i \subset U\}_{i \in I}$ (i.e. $\bigcup U_i = U$); all the conditions of a site are then satisfied. For instance, fiber products are given by intersections.

Definition 3. Given two sites $S_1 = (\mathcal{C}_1, \operatorname{Cov}(\mathcal{C}_1))$, $S_2 = (\mathcal{C}_2, \operatorname{Cov}(\mathcal{C}_2))$, a **morphism of sites**[2] $f : S_1 \to S_2$ is a functor $f : \mathcal{C}_1 \to \mathcal{C}_2$ such that

(i) $\{U_i \to U\}_{i \in I} \in \operatorname{Cov}(\mathcal{C}_1) \implies \{f(U_i) \to f(U)\}_{i \in I} \in \operatorname{Cov}(\mathcal{C}_2)$.

(ii) If $V \to U$ is any morphism in $\mathcal{C}_1$ and $\{U_i \to U\}_{i \in I} \in \operatorname{Cov}(\mathcal{C}_1)$, then for each $i \in I$, then the induced map $f(U_i \times_U V) \to f(U_i) \times_{f(U)} f(V)$ is an isomorphism.

f is moreover an **isomorphism** if f induces an equivalence of categories and if g is a quasi-inverse functor to f, then $\{V_i \to V\}_{i \in I} \in \operatorname{Cov}(\mathcal{C}_2) \implies \{g(V_i) \to g(V)\}_{i \in I} \in \operatorname{Cov}(\mathcal{C}_1)$.

We consider the standard definition of a topological space again.

Example 4. Suppose X and Y are topological spaces, in the standard sense. If $f : X \to Y$ is a continuous map, then we have the induced functor $\mathsf{Op}(Y) \to \mathsf{Op}(X)$ given by taking $f^{-1}(\cdot)$. This gives rise to a morphism of the corresponding sites.

We can now extend our standard definition of a sheaf on a topological space. To do so, we need to begin with the notion of a presheaf.

Definition 5. Suppose S is a site $(\mathcal{C}, \operatorname{Cov}(\mathcal{C}))$ and suppose $\mathcal{D}$ is a category that admits arbitrary products (including the empty product). A functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is called a **presheaf on the site** S **with values in** $\mathcal{D}$.[3] As usual, a morphism is given by natural transformations.

The sheaf condition for sites is similar to the standard case, except we replace intersections with fiber products. Noting that fiber products correspond to taking intersections, it's clear that this

[2]There are multiple definitions of a morphism of a sites; for instance, the Stacks Project calls our definition a **continuous morphism** instead.

[3]Note that a presheaf on a site only depends on data of the underlying category $\mathcal{C}$ and not its covering families; the extra data of $\operatorname{Cov}(\mathcal{C})$, however, comes up crucially in the definition of a sheaf.

is a generalization of the familiar definition.

Definition 6. A **sheaf on the site** S **with values in** $\mathcal{D}$ is a presheaf $\mathcal{F}$ that satisfies the following criterion:

For each covering family $\{U_i \to U\}_{i \in I}$, the diagram

$$\mathcal{F}(U) \hookrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} \mathcal{F}(U_i \times_U U_j)$$

is a exact, where the top and bottom maps, on each index, is induced by $\mathcal{F}$ applied to the projections $U_i \times_U U_j \to U_i$ and $U_i \times_U U_j \to U_j$, respectively (in particular, this means that the first map is injective and the kernel of the second map is equal to the image of the first map).

More specifically, we care about the situation where $\mathcal{D}$ is Ab, the category of abelian groups. Let $\mathcal{T}$ be a site and $\mathsf{PAb}(\mathcal{T})$ be the category of abelian presheaves on $\mathcal{T}$. Then,

(i) $\mathsf{PAb}(\mathcal{T})$ is an abelian category.

(ii) Any sequence $F \to G \to H$ in $\mathsf{PAb}(\mathcal{T})$ is exact iff the sequence $F(U) \to G(U) \to H(U)$ in Ab is exact for every $U \in \mathcal{T}$.

(iii) $\mathsf{PAb}(\mathcal{T})$ has enough injectives.

Moreover, there is a full subcategory of $\mathsf{PAb}(\mathcal{T})$ called $\mathsf{Ab}(\mathcal{T})$ comprising the abelian sheaves on $\mathcal{T}$. Let $i : \mathsf{Ab}(\mathcal{T}) \hookrightarrow \mathsf{PAb}(\mathcal{T})$ denote the inclusion. Then, there is a left adjoint functor of i called ${}^{\mathrm{ad}}i$. Moreover, for $F \in \mathsf{PAb}(\mathcal{T})$, we write $F^\sharp \in \mathsf{Ab}(\mathcal{T})$ to denote ${}^{\mathrm{ad}}i(F)$ and call it the **sheaf associated to** F, or the **sheafification of** F.

It turns out that $\mathsf{Ab}(\mathcal{T})$ is an abelian category with enough injectives, so that any left exact additive functor $f : \mathsf{Ab}(\mathcal{T}) \to \mathcal{C}$ (where $\mathcal{C}$ is any abelian category) has right derived functors $R^q f$.

In particular, if we fix some $U \in \mathcal{T}$, there is a section functor $\Gamma_U : \mathsf{Ab}(\mathcal{T}) \to \mathsf{Ab}$ that sends F to $F(U)$. The section functor is exact on the level of presheaves and the inclusion $\mathsf{Ab}(\mathcal{T}) \to \mathsf{PAb}(\mathcal{T})$ is left exact, so it follows that Γ_U has right derived functors.

Definition 7. Let $F \in \mathsf{Ab}(\mathcal{T})$. Then, we define the q**th cohomology group of** U **with values in** F by $H^q(U, F) = (R^q \Gamma_U)(F)$.

3 Étale site and operations on étale sheaves

Definition 8. Let X be a scheme. We let $\text{Ét}/X$ denote the category of étale X-schemes, which has finite fiber products (including the empty fiber product, namely that X is a final object).

A family $\{\varphi_i : X'_i \to X'\}$ in $\text{Ét}/X$ is said to be **surjective** if X' is covered by the images of φ_i. It's clear the set of all surjective families satisfies the necessary axioms of a site, and so we define $X_{\text{ét}}$ (or the **étale site of** X) to be the site whose objects are $\text{Ét}/X$ and whose coverings are the surjective families in $\text{Ét}/X$.

Remark 9. By the Leray spectral sequence, we can compare the Zariski and étale sites. Here, the Zariski site refers to the topology of open sets of a scheme X (morphisms are given by inclusion). Note that there is an inclusion $i : X_{\text{Zar}} \to X_{\text{ét}}$, which is a morphism of sites. Then, we have the spectral sequence

$$E_2^{p,q} = H^p_{\text{Zar}}(X, R^q i^s(F)) \Rightarrow E^{p+q} = H^{p+q}_{\text{ét}}(X, F).$$

Now, let $f : X \to Y$ be a morphism of schemes. Then, f induces a covariant functor from $\text{Ét}/Y$ to $\text{Ét}/X$ by sending $Y' \mapsto Y' \times_Y X$, which respects fiber products and surjective families of morphisms, so that we get a morphism of sites $f_{\text{ét}} : Y_{\text{ét}} \to X_{\text{ét}}$. As a result, we can define

$$f_* \coloneqq (f_{\text{ét}})^s : \mathsf{Ab}(X_{\text{ét}}) \to \mathsf{Ab}(Y_{\text{ét}}) \text{ and } f^* \coloneqq (f_{\text{ét}})_s : \mathsf{Ab}(Y_{\text{ét}}) \to \mathsf{Ab}(X_{\text{ét}}).$$

The former is called the **direct image of** F and the latter the **inverse image of** G. Note that $f_*F(Y') = F(Y' \times_Y X)$. and that $f^*G(X')$ is the sheaf associated to the presheaf $\varinjlim_{\mathcal{I}_{X'}^{\text{op}}} G$, where $\mathcal{I}_{X'}$ is the category of pairs (Y', φ) with $Y' \in \text{Ét}/Y$ and $\varphi : X' \to Y' \times_Y X$ an X-morphism. But, note that $\operatorname{Hom}_X(X', Y' \times_Y X) = \operatorname{Hom}_Y(X', Y')$, so we can also view $\mathcal{I}_{X'}$ as the category of Y-morphisms $X' \to Y'$ to $Y' \in \text{Ét}/Y$.

Remark 10. $\mathcal{I}_{X'}^{\text{op}}$ is filtered, since $f_{\text{ét}}$ preserves fiber products and final objects.

Proposition 11.

(i) f^* *is left adjoint to* f_*.

(ii) f_* *is left exact.*

(iii) f^* *is exact and commutes with inductive limits.*

(iv) $(g \circ f)_* = g_* \circ f_*$ *and* $(g \circ f)^* = f^* \circ g^*$.

Since f_* is left exact, we have the usual right derived functors $R^q f_*$, which correspond to the sheaves associated to the presheaves $H^q(- \times_Y X, F)$ on $Y_{\text{ét}}$. As a result, we have the following from the Leray spectral sequence.

Proposition 12. *Let* $F \in \mathsf{Ab}(X_{\text{ét}})$ *and* $Y' \in \text{Ét}/Y$. *Then, if* $f : X \to Y$ *is a morphism of schemes, we have the spectral sequence*

$$E_2^{p,q} = H^p(Y', R^q f_*(F)) \Rightarrow E^{p+q} = H^{p+q}(Y' \times_Y X, F).$$

More generally, we have

Proposition 13. *If* $f : X \to Y$ *and* $g : Y \to Z$ *are morphisms of schemes and* $F \in \mathsf{Ab}(X_{\text{ét}})$, *we have the spectral sequence*

$$E_2^{p,q} = R^p g_*(R^q f_*(F)) \Rightarrow E^{p+q} = R^{p+q}(gf)_*(F).$$

From this spectral sequence, we obtain the edge morphism $E_2^{p,0} \to E^p$, namely

$$H^p(Y', f_*F) \to H^p(Y' \times_Y X, F),$$

which is functorial in F. Also, we have the unit $G \to f_* f^* G$ inducing $H^p Y', G) \to H^p(Y', f_* f^* G)$, along with edge morphism $H^p(Y', f_* f^* G) \to H^p(Y' \times_Y X, f^* G)$, whose composition gives

$$H^p(Y', G) \to H^p(Y' \times_Y X, f^* G),$$

which is functorial in G.

We can also construct the **base-change morphisms** using the Leray spectral sequence.

First, note that we have edge morphisms

$$R^p g_*(f_* F) \to R^p(gf)_*(F) \text{ and } R^p(gf)_*(F) \to g_*(R^p f_*(F)).$$

Now, suppose we have a Cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{f} & Y \end{array}$$

of morphisms of schemes. If $F \in \mathsf{Ab}(X_{\text{ét}})$, we can consider the unit $F \to v'_* v'^* F$, which along with the edge morphisms, induces the maps

$$\begin{aligned} R^p f_*(F) &\to R^p f_*(v'_* v'^* F) \\ &\to R^p(f \circ v')_*(v'^* F) \\ &= R^p(v \circ f')_*(v'^* F) \\ &\to v_*(R^p f_*(v'^* F)), \end{aligned}$$

which induces a map

$$v^*(R^p f_*(F)) \to R^p f'_*(v'^* F))$$

called the base-change morphism, functorial in F. We will see in the future under what conditions this is an isomorphism.

At this point, we should at least mention a couple examples of étale sheaves, although we might not be able to talk about them further.

Remark 14. $X_{\text{ét}}$ is subcanonical in the sense that every covering in $X_{\text{ét}}$ is a family of universal effective epimorphism in the category of X-schemes. In other words, every representable presheaf of sets (with representing object in the category of X-schemes) is a sheaf on $X_{\text{ét}}$.

Now, given a commutative group scheme over X, we write G_X to denote the sheaf on $X_{\text{ét}}$ represented by G—this is doable because of our remark above. G_X is a sheaf of abelian groups on $X_{\text{ét}}$ with $G_X(X') = \mathrm{Hom}_X(X', G)$.

Definition 15. The following four commutative group schemes are important for the study of Artin-Schreier and Kummer theory. Let X be a scheme and $X' \in \text{Ét}/X$.

(i) The **additive group** $(\mathbb{G}_a)_X$ is defined so that $\mathbb{G}_a = \mathrm{Spec}(\mathbb{Z}[t]) \times_{\mathrm{Spec}\,\mathbb{Z}} X$. The points of $\mathbb{G}_a$

are as follows:

$$\begin{aligned}(\mathbb{G}_a)_X(X') &= \mathrm{Hom}_X(X', \mathrm{Spec}(\mathbb{Z}[t]) \times_{\mathrm{Spec}\,\mathbb{Z}} X)\\ &= \mathrm{Hom}(X', \mathrm{Spec}(\mathbb{Z}[t]))\\ &= \mathrm{Hom}(\mathbb{Z}[t], \Gamma(X', \mathcal{O}_{X'}))\\ &= \Gamma(X', \mathcal{O}_{X'}).\end{aligned}$$

(ii) The **multiplicative group** $(\mathbb{G}_m)_X$ is defined so that $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t, t^{-1}]) \times_{\mathrm{Spec}\,\mathbb{Z}} X$. The points of $\mathbb{G}_m$ are as follows:

$$\begin{aligned}(\mathbb{G}_m)_X(X') &= \mathrm{Hom}(\mathbb{Z}[t, t^{-1}], \Gamma(X', \mathcal{O}_{X'}))\\ &= \Gamma(X', \mathcal{O}_{X'})^{\times}.\end{aligned}$$

(iii) The n**th roots of unity** $(\mu_n)_X$ is defined so that $\mu_n = \mathrm{Spec}\left(\mathbb{Z}[t]/(t^n - 1)\right) \times_{\mathrm{Spec}\,\mathbb{Z}} X$. The points of μ_n are as follows:

$$\begin{aligned}(\mu_n)_X(X') &= \mathrm{Hom}\left(\mathbb{Z}[t]/(t^n - 1), \Gamma(X', \mathcal{O}_{X'})\right)\\ &= \left\{r \in \Gamma(X', \mathcal{O}_{X'}) : r^n = 1\right\}.\end{aligned}$$

(iv) The **constant sheaf** A_X **associated to a abelian group** A is defined so that A_X is the constant sheaf associated to A. In particular, $A_X(X')$ is the set of continuous functions from X' to A (endowed with the discrete topology), which is the set of locally constant functions from X' to A, which corresponds to partitions of X' into a disjoint union of open subsets, indexed by elements of A. So

$$A_X(X') = \mathrm{Hom}_X\left(X', \coprod_A X\right).$$

Moreover, $\coprod_A X$ is indeed an étale group scheme over X (with group structure induced by A).

4 Frobenius action

Before discussing the Frobenius endomorphism on a scheme, we recall some general facts about functoriality. Let $f : X' \to X$ and $F \in \mathsf{Ab}(X_{\text{ét}})$. Note that we have a natural map $H^i(X, F) \to H^i(X', f^*F)$, defined by the composition $H^i(X, F) \to H^i(X, f_*f^*F) \to H^i(X', f^*F)$, induced by the adjunction and the edge morphism from the Leray spectral sequence for f^*F.

Definition 16. Suppose X is a scheme over $\mathbb{F}_q$. We define $\mathrm{Fr} : X \to X$, the **absolute Frobenius**, to be the morphism defined as the identity on $|X|$ and the qth power map on the level of structure sheaves.

Lemma 17. *There is a canonical morphism of sheaves* $\mathrm{Fr}^*_{/X} : F \to \mathrm{Fr}_* F$*, which is an isomorphism.*

Proof. Let $f : U \to X$ be étale and note that $f \circ \mathrm{Fr} = \mathrm{Fr} \circ f$. This induces a unique X-morphism $\mathrm{Fr}_{U/X} : U \to X \times_X U$, where $X \to X$ is Fr. Note that the composition $U \to X \times_X U \to X$ is just $U \to X$, and since both $U \to X$ and $X \times_X U \to X$ are étale (by base change), it follows that $\mathrm{Fr}_{U/X}$ is étale. We also see that it is universally bijective, so it follows that it is an isomorphism. The result follows. ☺

Remark 18. By adjunction, we get a corresponding morphism $\mathrm{Fr}^* F \to F$.

Lemma 19. *The induced homomorphism on cohomology $H^i(X, F) \to H^i(X, \mathrm{Fr}^* F) \to H^i(X, F)$ is the identity.*

One might ask what happens when we base change to the algebraic closure. In this setting, we actually get many different Frobenii:

Definition 20. Let $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$.

(i) The **absolute Frobenius** on $\overline{X}$ is simply $\mathrm{Fr}_{\overline{X}}$ (here, we write the base scheme in the subscript to avoid ambiguity).

(ii) The **relative Frobenius** on $\overline{X}$ is $\mathrm{Fr}_X \times \mathrm{Id}$.

(iii) The **arithmetic Frobenius** on $\overline{X}$ is $\mathrm{Id} \times \mathrm{Fr}_{\overline{\mathbb{F}_q}}$.

(iv) The **geometric Frobenius** on $\overline{X}$ is $\mathrm{Id} \times \mathrm{Fr}_{\overline{\mathbb{F}_q}}^{-1}$.

A little more concretely, the relative Frobenius corresponds to raising variables to the qth power, the arithmetic Frobenius corresponds to raising coefficients to the qth power, the geometric Frobenius corresponds to taking pth roots of coefficients, and the absolute Frobenius corresponds to raising everything to the pth power.

We note that $(\mathrm{Fr}_X \times \mathrm{Id}) \circ (\mathrm{Id} \times \mathrm{Fr}_{\overline{\mathbb{F}_q}}) = \mathrm{Fr}_X \times \mathrm{Fr}_{\overline{\mathbb{F}_q}} = (\mathrm{Id} \times \mathrm{Fr}_{\overline{\mathbb{F}_q}}) \circ (\mathrm{Fr}_X \times \mathrm{Id})$ is the absolute Frobenius $\mathrm{Fr}_{\overline{X}}$, and since the absolute Frobenius induces the identity on the level of cohomology, it follows that relative Frobenius and geometric Frobenius induce the same action on cohomology.

5 Stalks of étale sheaves

Definition 21. A **geometric point of a scheme** X is an X-scheme of the form $P = \mathrm{Spec}(\Omega)$, where Ω is a separably closed field. In other words, we are specifying a point $x \in X$ together with $\kappa(x) \hookrightarrow \Omega$.

Remark 22. $F \mapsto F(P)$ actually gives an equivalence between $\mathsf{Ab}(P_{\text{ét}})$ and Ab.

Definition 23. Let $u : P \to X$ be a geometric point of X. Then, if $F \in \mathsf{Ab}(X_{\text{ét}})$, we say that $F_p := (u^* F)(P)$ is the **stalk of F in P**.

Example 24. Let G be an étale commutative group scheme over X and let G_X be the abelian sheaf on $X_{\text{ét}}$ represented by G. Then, note that $u^* G_X(P) = \mathrm{Hom}_X(P, G)$ is just the Ω-valued points of G. In particular, the stalk of the constant sheaf A_X is just A.

Proposition 25.

(i) $F \mapsto F_P$ is exact and commutes with inductive limits.

(ii) If $v : P' \to P$ is an X-morphism of geometric points of X, then $F_p \cong F_{P'}$.

(iii) *Consider $f : X \to Y$ and P a geometric point of X. Then, P is a geometric point of Y (via f) and for any $F \in \mathsf{Ab}(Y_{ét})$, we have $(f^* F)_P \cong F_P$.*

Remark 26. For any $x \in X$, consider $\kappa(x)$ and a separable closure $\overline{\kappa(x)}$. Then, let $\overline{x}$ be the geometric point corresponding to $\operatorname{Spec}\left(\overline{\kappa(x)}\right) \to \operatorname{Spec}(\kappa(x)) \to X$. Then, the proposition above tells us that every stalk F_P where the image of P is $x \in X$ is the same as $F_{\overline{x}}$ (noting that separable closures are (non-uniquely) isomorphic).

Definition 27. An **étale neighborhood of P in X** is a pair (X', u') with $X' \in \text{Ét}/X$ and $u' : P \to X'$ and X-morphism.

Remark 28. The dual category of étale neighborhoods of P in X is filtered and we get a canonical homomorphism

$$\varinjlim_{X'} F(X') \to F_P,$$

by noting that the former comes from the description of the presheaf $u^{\cdot} F(P)$ and the latter the sheafification. Let the image of $s \in F(X')$ in F_P be written s_P.

Lemma 29. *Let $f : X \to Y$ and $G \in \mathsf{PAb}(Y_{ét})$. Then, $(f^{\cdot} G)^{\#} \to f^*(G^{\#})$ (induced by $f^{\cdot} G \to f^{\cdot}(G^{\#}) \to f^*(G^{\#})$), is an isomorphism.*

Proof. In general, if we have a morphism of topologies $f : \mathcal{T} \to \mathcal{T}'$ with G an abelian presheaf on $\mathcal{T}$, then $(f_p G)^{\#} \to f_s(G^{\#})$ is an isomorphism by using the push-pull adjunction twice. ☺

Proposition 30. *Let G be an abelian presheaf on $X_{ét}$, where P is a geometric point of X. Then,*

$$\varinjlim_{X'} G(X') \to (G^{\#})_P$$

is an isomorphism.

Proof sketch. It suffices to show that $G(P) \to G^{\#}(P)$ is an isomorphism. To do this, we note that $\{\mathrm{Id} : P \to P\}$ is always a refinement of any $\{U_i \to P\}$. ☺

Example 31. Let $f : X \to Y$ and $F \in \mathsf{Ab}(X_{ét})$, with P a geometric point of Y. Then,

$$R^q f_*(F)_P \cong \varinjlim_{Y'} H^q(X \times_Y Y', F),$$

where the inductive limit goes over the étale neighborhoods of P in Y.

Example 32. If k is a field and k^{sep} a separable closure, then $F \mapsto \varinjlim_{k'} F(\operatorname{Spec}(k'))$ (running over finite subextensions) is an equivalence between $\mathsf{Ab}(\operatorname{Spec}(k)_{ét})$ and continuous $\operatorname{Gal}(k^{\text{sep}}/k)$-modules.

$\operatorname{Spec}(k^{\text{sep}})$ is a geometric point of $\operatorname{Spec}(k)$ (induced by the inclusion), and there is a full subcategory of the étale neighborhoods given by finite subextensions k' of k^{sep}/k, which is clearly initial, so that in the dual category we get a final subcategory. Then, we have

$$\varinjlim_{k'} F(\operatorname{Spec}(k')) \cong F_{\operatorname{Spec}(k^{\text{sep}})}.$$

Theorem 33.

(i) The morphism $F' \to F$ of abelian sheaves on $X_{ét}$ is an isomorphism/monomorphism/epimorphism iff the morphism $F'_{\overline{x}} \to F_{\overline{x}}$ is for every $x \in X$.

(ii) The morphism $v : F' \to F$ of abelian sheaves on $X_{ét}$ is 0 iff $v_{\overline{x}} = 0$ for all $x \in X$. In particular, for $s \in F(X)$, we have $s = 0$ iff $s_{\overline{x}} = 0$ for all $x \in X$.

(iii) $F' \to F \to F''$ in $\mathsf{Ab}(X_{ét})$ is exact iff $F'_{\overline{x}} \to F_{\overline{x}} \to F''_{\overline{x}}$ is for every $x \in X$.

Proof. Sketch: Most of these statements follow easily from showing the statement about isomorphisms in part 1. To do this, the key point (and in similar vein to the usual proof about stalks) is noting that the dual category of étale neighborhoods of $\overline{x}$ is filtered, so that we can find étale neighborhoods X' of $\overline{x}$ where we can find vanishing s on X' for every $s_{\overline{x}} = 0$. ☺

Remark 34. More generally, any finite (co)limits can be checked at the level of points.

6 Cohomology with compact support

Proposition 35. *Let $\mathcal{T}$ be a topology and $F \in \mathsf{Ab}(\mathcal{T})$. Then, F is a sheaf associated to a presheaf of abelian torsion groups iff the canonical morphism $\varinjlim {}_nF \to F$ is actually an isomorphism. Here, ${}_nF$ is the kernel of the multiplication-by-n map $F \xrightarrow{n} F$ (with $n \in \mathbb{N}$).*

If F satisfies either of these conditions, we say that F is a ***torsion sheaf****.*

Remark 36. In general, for a torsion sheaf F, it is not the case that $F(U)$ are all torsion groups. However, if we further ask that U is quasicompact, then $F(U)$ is actually torsion. This follows from the quasicompactness of U (which allows to work with finite covers and the fact that the inductive limit presheaf P is already separated):

$$\begin{aligned} F(U) &= P^{\dagger}(U) \\ &= \check{H}^0(U, P) \\ &= \varinjlim_{\{U_i \to U\} \text{ finite}} H^0(\{U_i \to U\}, P) \\ &= \varinjlim_{\{U_i \to U\} \text{ finite}} \ker\left(\prod_i P(U_i) \rightrightarrows \prod_{i,j} P(U_i \times_U U_j)\right) \end{aligned}$$

is clearly torsion.

Remark 37. The following facts may be useful:

(i) Let $F \in \mathsf{Ab}(X_{ét})$. Then, F is torsion iff $F_{\overline{x}}$ is torsion for every $x \in X$.

(ii) Let X be qcqs and $F \in \mathsf{Ab}(X_{ét})$ is torsion. Then, $H^q(X, F)$ are torsion for every $q \geq 0$.

(iii) Let $f : X \to Y$ and $F \in \mathsf{Ab}(Y_{ét})$ is torsion. Then, $f^*F \in \mathsf{Ab}(X_{ét})$ is torsion. Also, if f is qcqs and instead $F \in \mathsf{Ab}(X_{ét})$ is torsion. Then, $R^q f_* F \in \mathsf{Ab}(Y_{ét})$ is torsion for all q.

Example 38. Examples of torsion sheaves include $(\mu_n)_X$ and constant sheaves A_X with A a discrete abelian torsion group.

Definition 39. $F \in \mathsf{Ab}(X_{\text{ét}})$ is called **locally constant** if there is a covering $\{X_i \to X\}$ so that $F|_{X_i}$ is constant.

$F \in \mathsf{Ab}(X_{\text{ét}})$ is **finite** if every stalk $F_{\overline{x}}$ is finite.

$F \in \mathsf{Ab}(X_{\text{ét}})$ is called **constructible** if each affine open subset $U \subset X$ can be written as a union of finitely many constructible reduced subschemes U_i of U so that $F|_{U_i}$ is locally constant and finite.

For the rest of this section, suppose X is separated of finite type over a field k.

Given an open immersion $j : U \subset X$, we can define an extension-by-zero functor $j_! : \mathsf{Ab}(U_{\text{ét}}) \to \mathsf{Ab}(X_{\text{ét}})$ that satisfies the property that for any geometric point $\overline{x}$ of X and $F \in \mathsf{Ab}(X_{\text{ét}})$, we have

$$(j_! F)_{\overline{x}} = \begin{cases} F_{\overline{x}} & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases}$$

Remark 40. One way to do this is via the decomposition theorem: suppose we have a closed immersion $Y \hookrightarrow X$ with complement $U \subset X$. Then, there is an equivalence of categories between $\mathsf{Ab}(X_{\text{ét}})$ and $T(X, Y)$, the mapping cylinder category associated to X and Y.

We can then define $H^i_c(U, F) = H^i(X, j_! F)$, where X is proper over k and contains U as a dense open subset.

Proposition 41. *Let F be a torsion sheaf on X.*

(i) $H^i_c(X, F)$ exists and makes sense.

(ii) $F \mapsto H^i_c(X, F)$ is an exact δ-functor.

(iii) If $i : Z \hookrightarrow X$ is a closed immersion and $j : U \subset X$ is the open immersion defined with $U = X - Z$, then we have a long exact sequence

$$\cdots \to H^{i-1}_c(Z, i^* F) \to H^i_c(U, j^* F) \to H^i_c(X, F) \to H^i_c(Z, i^* F) \to H^{i+1}_c(U, j^* F) \to \cdots.$$

Before we get to the proof, we cite the proper base change, a deep and technical theorem:

Theorem 42 (Proper base change)**.** *Let $f : X \to Y$ be a proper morphism of schemes and F be a torsion sheaf.*

(i) If F is a constructible sheaf on X, then $R^i f_* F$ is constructible for all $i \geq 0$.

(ii) For any Cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{f} & Y \end{array}$$

the base change morphism $g^*(R^i f_* F) \to R^i f'_*(g'^* F)$ is an isomorphism for every i.

(iii) Let $\overline{y} \in Y$ be a geometric point and $\pi : X_{\overline{y}} \to X$. Then, we have canonical isomorphisms

$$(R^i f_* F)_{\overline{y}} \cong H^i(X_{\overline{y}}, \pi^* F).$$

Proof sketch of proposition. The tricky part is (i); we note that (ii) and (iii) follow easily once one notes that $\mu_!$ is exact. By Nagata's compactification theorem, we know that there exists some open immersion $\mu : X \subset X_1$ with X_1 a proper k-scheme and X dense in X_1. We want to check that the choice of compatification does not matter.

So suppose $\nu : X \subset X_2$ is another compactification. We can consider the scheme-theoretic image of X in $X_1 \times X_2$ as another compatification, so that we can assume WLOG that there is a morphism $h : X_1 \to X_2$ so that $h \circ \mu = \nu$. Now, the Leray spectral sequence tells us that

$$H^p(X_2, R^q h_* \mu_! F) \Rightarrow H^{p+q}(X_1, \mu_! F).$$

Then, if we can show that $R^q h_* \mu_! F = \nu_! F$ for $q = 0$ and 0 otherwise, we are done. We can check this easily on the level of stalks by invoking the proper base change theorem, noting that $(R^q h_* \mu_! F)_{\overline{x}} = H^q\left((X_1)_{\overline{x}}, (\mu_! F|_{X_1})_{\overline{x}}\right) = F_{\overline{x}}$ for $q = 0$ and $x \in X$ and 0 otherwise. ☺

7 Important theorems and the necessity of torsion coefficients

Now that we have seen some properties of étale cohomology, it may be tempting to assume that the scheme-theoretic analogue of singular cohomology (with $\mathbb{Z}$-coefficients) is simply taking étale cohomology of the constant sheaf $\mathbb{Z}_X$. Unfortunately, this does not work.

Proposition 43. *Let X be regular. Then, $H^1(X_{ét}, \mathbb{Z}_X) = 0$.*

Remark 44. The main issue here is that $\mathbb{Z}$ does not have any torsion!

As a result, any sensible analogue of singular cohomology will need to consider torsion sheaves. One way to do this is as follows.[4]

Definition 45. Let X be a k-scheme and l a prime different from $\mathrm{char}(k)$. Then, define

$$H^i(X_{\text{ét}}, \mathbb{Z}_l) = \varinjlim_n H^i(X_{\text{ét}}, \mathbb{Z}/l^n\mathbb{Z}),$$

which is a $\mathbb{Z}_l$–module. We can then define

$$H^i(X_{\text{ét}}, \mathbb{Q}_l) = H^i(X_{\text{ét}}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Theorem 46 (Cohomological dimension)**.** *If X is affine and finite type over a separably closed field L, then $H^i(X, \mathbb{Q}_l) = 0$ for $i > \dim X$.*

Theorem 47 (Poincaré duality)**.** *Let X be smooth and connected of dimension d, and separated over a separably closed field k. Also, let l be coprime to $\mathrm{char}(k)$. Then, there is a trace map isomorphism $\mathrm{tr} : H_c^{2d}(X, \mathbb{Q}_l)(d) \to \mathbb{Q}_l(-d)$.[5] Also, there is a perfect pairing*

$$H_c^i(X, \mathbb{Q}_l) \times H^{2d-i}(X, \mathbb{Q}_l)(d) \to H^{2d}(X, \mathbb{Q}_l)(d) \xrightarrow{\mathrm{tr}} \mathbb{Q}_l(-d).$$

Theorem 48 (Weak Lefschetz)**.** *Let X be smooth projective of dimension d over a separably closed field k, and let $Y \hookrightarrow X$ be a smooth hyperplane section. Then, the restriction map $H^i(X, \mathbb{Q}_l) \to H^i(Y, \mathbb{Q}_l)$ is bijective for $0 \le i \le d-2$ and injective for $i = d-1$.*

[4]This notation is a bit confusing; these are not the étale cohomology groups of the constant sheaf $\mathbb{Z}_l$ or $\mathbb{Q}_l$!

[5]I haven't defined what "(d)" means, but they are essentially the same as Tate twists.

Proof. Let $U = X - Y$, which we note is affine open in X, since $X \hookrightarrow \mathbb{P}^N$ is affine and for any hyperplane $H \hookrightarrow \mathbb{P}^N$, we have $\mathbb{P}^N - H$ is affine. From the two theorems above, we have that $H_c^i(U, \mathbb{Q}_l) = 0$ for $i < \dim X$.

Using the long exact sequence of compact cohomology groups, and noting that for X and Y, compact cohomology is the usual cohomology, the result follows. ☺

Remark 49. I haven't stated these in full generality, but I expect the next talk will talk again about l-adic sheaves anyways, so this is just a glimpse of what's next.

8 Appendix

This section isn't very important and is mainly just a reminder for the author on some facts about abelian categories and spectral sequences.

8.1 Abelian categories

Definition 50. A category $\mathcal{C}$ is **additive** if it has the following properties.

(i) For any $A, B \in \mathcal{C}$, the set $\mathrm{Hom}(A, B)$ has the structure of an abelian group and composition of morphisms is bilinear.

(ii) Finite products and sums exist.

(iii) There is a zero object.

Definition 51. A category $\mathcal{C}$ is **abelian** if it has the following properties.

(i) Every morphism has a kernel and a cokernel.

(ii) For every morphism u, the canonical map $\mathrm{coim}(u) \to \mathrm{im}(u)$ is an isomorphism.

Proposition 52. *In an abelian category, any bijective morphism is an isomorphism.*

Proposition 53. *Let $\mathcal{C}$ and $\mathcal{C}'$ be categories with $\mathcal{C}'$ abelian. Then, the category of natural transformations $\mathrm{Hom}(\mathcal{C}, \mathcal{C}')$ is an abelian category. Also, any $F \to G \to H$ in $\mathrm{Hom}(\mathcal{C}, \mathcal{C}')$ is exact iff $F(A) \to G(A) \to H(A)$ is exact for all $A \in \mathcal{C}$.*

Definition 54. Let $\mathcal{C}$ be an abelian category. Then, $M \in \mathcal{C}$ is **injective** iff $\mathrm{Hom}(-, M)$ is exact. $\mathcal{C}$ is said to **have enough injectives** if for any object $A \in \mathcal{C}$, there is a monomorphism $A \to M$ for some injective object M.

Definition 55. A family $(Z_i)_{i \in I}$ of objects is called a **family of generators** if for any A and B a proper subobject of A, there is some Z_i and a morphism $Z_i \to A$ that does not factor through the the inclusion $B \to A$.

Condition 56. There are two important properties for an abelian category $\mathcal{C}$ that we call **AB 3)** and **AB 5)**.

- If arbitrary direct sums exist, we say that AB 3) is satisfied.
- If AB 3) is satisfied and moreover for any increasingly filtered family of subobjects A_i of A, and given morphisms $u_i : A_i \to B$ (for some fixed B), such that the u_i are induced by

restriction, there is a unique morphism $u : \sum_i A_i \coloneqq \mathrm{im}(\bigoplus_i A_i \to A) \to B$ extending all of the u_i, we say AB 5) is satisfied.

Proposition 57. *Let $\mathcal{C}$ be an abelian category satisfying AB 3) with $(Z_i)_{i\in I}$ a family of objects and Z the direct sum of the Z_i. Then, TFAE:*

(i) $(Z_i)_{i\in I}$ is a family of generators of $\mathcal{C}$.

(ii) Z is a generator of $\mathcal{C}$.

(iii) There is an epimorphism $\bigoplus_{j\in J} Z \to A$ for any $A \in \mathcal{C}$.

Proposition 58. *Let $\mathcal{C}$ be an abelian category satisfying AB 5) that also has a family of generators. Then, $\mathcal{C}$ has enough injectives.*

Example 59. The category Ab (of abelian groups) has generator $\mathbb{Z}$ and satisfies AB 5). Hence, Ab has enough injectives. Of course, we can also prove this directly.

Proposition 60. *Let $\mathcal{C}$ and $\mathcal{C}'$ be categories with $\mathcal{C}'$ abelian. If C' satisfies Ab 5), then so does $\mathrm{Hom}(\mathcal{C}, \mathcal{C}')$. If $\mathcal{C}'$ has generators and satisfies AB 3), then so does $\mathrm{Hom}(\mathcal{C}, \mathcal{C}')$.*

Lemma 61. *Suppose $\mathcal{A}, \mathcal{B}$ are abelian categories with $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{A}$ covariant functors that are adjoint to each other. Suppose F is the right adjoint and that G is exact. Then, F sends injectives to injectives.*

Proof. If I is an injective object, we want to show that $\mathrm{Hom}(-, FI)$ is injective. We can then take short exact sequence, hit it with G to get an exact sequence, and then hit it with $\mathrm{Hom}(-, I)$, which is also exact by definition, and then use the adjunction between F and G to get the desired result. ☺

8.2 Spectral sequences

For this section, fix an abelian category $\mathcal{C}$.

Definition 62. For each $r \geq 0$, a **spectral sequence** consists of objects $\{E_r^{p,q}\}_{p,q\in\mathbb{Z}}$ and differentials $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ so that $E_{r+1}^{p,q} = \ker(E_r^{p,q} \to E_r^{p+r,q-r+1}) / \mathrm{im}(E_r^{p-r,q+r-1} \to E_r^{p,q})$. A spectral sequence is said to live in the **first quadrant** if $p < 0$ or $q < 0$ implies $E_r^{p,q} = 0$.

If $K^\cdot = F^0K^\cdot \supset F^1K^\cdot \supset \cdots$ is a filtered complex, we write $\mathrm{gr}^p K^\cdot$ to express the quotient $F^pK^\cdot / F^{p+1}K^\cdot$. Moreover, there is a natural filtration of $H(K^\cdot)$.

Theorem 63. *Suppose $K^\cdot$ is a nonnegative filtered complex, i.e. a filtered complex such that $K^n = 0$ for negative n). Then, there is an associated spectral sequence $E_r^{p,q}$ so that $E_0^{p,q} = \mathrm{gr}^p K^{p+q}$, $E_1^{p,q} = H^{p+q}(\mathrm{gr}^p K^\cdot)$, and for sufficiently large r, $E_r^{p,q} = \mathrm{gr}^p H^{p+q}(K^\cdot)$.*

We can rephrase the last part of the theorem in terms of convergence.

Definition 64. Let $E_r^{p,q}$ be a spectral sequence and suppose that for any pair (p, q), the term $E_r^{p,q}$ stabilizes as r becomes sufficiently large. We denote this term by $E_\infty^{p,q}$. Moreover, suppose we have a collection of objects $\{H^n\}_n$ with finite filtrations, i.e. the length of the chain of the filtration is finite. Then, we say $E_r^{p,q}$ converges to $H^\cdot$, written as $E_r^{p,q} \Rightarrow H^{p+q}$, if $E_\infty^{p,q} = \mathrm{gr}^p H^{p+q} = F^pH^{p+q} / F^{p+1}H^{p+q}$.

Remark 65. In the theorem above, we are saying that $E_1^{p,q} = H^{p+q}(\mathrm{gr}^p K^\cdot) \Rightarrow H^{p+q}(K^\cdot)$.

Lemma 66. *Suppose* $E_r^{p,q} \Rightarrow H^{p+q}$. *Then, if* $E_\infty^{p,q} = 0$ *unless* $q = q'$, *then* $H^n = E_\infty^{n-q',q'}$. *A similar statement holds for the first coordinate.*

Lemma 67. *Suppose* $E_r^{p,q} \Rightarrow H^{p+q}$. *If* $E_r^{p,q}$ *is a first quarter spectral sequence, then* H^n *has a filtration of the form* $0 = F^{n+1}H^n \subset F^n H^n \subset \cdots \subset F^0 H^n = H^n$. *A similar claim holds for third quarter spectral sequences.*

The main theorem that makes spectral sequences so useful is the application to double complexes.

Definition 68. A **double complex** M comprises the data of a bigraded object $M = \bigoplus_{p,q\in\mathbb{Z}} M^{p,q}$ and horizontal and vertical differentials $d : M^{p,q} \to M^{p+1,q}$ and $\delta : M^{p,q} \to M^{p,q+1}$ so that $d^2 = \delta^2 = d\delta + \delta d = 0$. We can associate to M a single complex called the **total complex**, defined as $\operatorname{Tot}^n M = \bigoplus_{p+q=n} M^{p,q}$. Its differential is given by $D = d + \delta$ (noting that $(d+\delta)^2$ is indeed 0).

There are two obvious filtrations we can impose on the total complex, given by horizontal "splicing" and vertical "splicing."

Definition 69. Suppose $(\operatorname{Tot} M, D)$ is a total complex of a double complex M. Then, ${}'F^p \operatorname{Tot}^n M = \bigoplus_{r+s=n, r\geq p} M^{r,s}$ and ${}''F^q \operatorname{Tot}^n M = \bigoplus_{r+s=n, s\geq q} M^{r,s}$ are two filtrations.

Theorem 70. *Suppose* $(\operatorname{Tot} M, D)$ *is a total complex of a double complex* M. *Then, the two spectral sequences* ${}'E_r^{p,q}$ *and* ${}''E_r^{p,q}$ *associated to the obvious filtrations satisfy the following properties:*

(i) ${}'E_0^{p,q} = M^{p,q}, {}''E_0^{p,q} = M^{q,p}$.

(ii) ${}'E_1^{p,q} = H_\delta^q(M^{p,\cdot}), {}''E_1^{p,q} = H_d^q(M^{\cdot,p})$.

(iii) ${}'E_2^{p,q} = H_d^p(H_\delta^q(M)), {}''E_2^{p,q} = H_\delta^p(H_d^q(M))$.

Furthermore, if M *is either a first or third quadrant double complex, then* ${}'E_r^{p,q}, {}''E_r^{p,q}$ *converge to* $H^{p+q}(\operatorname{Tot} M)$.

Proposition 71. *Let* $E_2^{p,q} \Rightarrow H^{p+q}$ *be a first quadrant spectral sequence. Then, there is an injection* $E_\infty^{n,0} \hookrightarrow H^n$. *For* $r \geq 2$, *the differential out of* $E_r^{n,0}$ *is the zero map, and so we have surjections* $E_r^{n,0} \twoheadrightarrow E_{r+1}^{n,0} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{n,0}$, *which compose to give us a map* $E_r^{n,0} \to H^n$. *Similarly, we get a map* $H^n \twoheadrightarrow E_\infty^{0,n} \hookrightarrow E_r^{0,n}$. *These maps are called the* ***edge maps****. Then, the sequence (given by the edge/obvious maps)*

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \to E_2^{2,0} \to H^2$$

is exact.

There is also interaction between derived functors and spectral sequences, expressed through the Grothendieck spectral sequence.

Theorem 72 (Grothendieck spectral sequence)**.** *Let* $\mathcal{A}, \mathcal{B}, \mathcal{C}$ *be abelian categories with enough injectives and suppose we have functors* $G : \mathcal{A} \to \mathcal{B}$ *and* $\mathcal{F} : \mathcal{B} \to \mathcal{C}$ *that are left exact and covariant. Moreover, suppose* GI *is* F*-acyclic for every injective objective* I *of* $\mathcal{A}$. *Then, for every* $A \in \mathcal{A}$, *there is a spectral sequence*

$$E_2^{p,q} = (R^p F)(R^q G)(A) \Rightarrow R^{p+q}(FG)(A).$$

Before we get into the proof, we state the following lemma.

Lemma 73. *Suppose* $\mathcal{A}$ *is an abelian category with enough injectives. Then, any complex* $C^\cdot$ *in* $\mathcal{A}$ *has a fully injective resolution, i.e. an injective resolution* $I^{\cdot,\cdot}$ *of* $C^\cdot$ *(fixing the first coordinate gives an injective resolution) so that the induced complexes* $Z^p(C^\cdot) \to Z^p(I^{\cdot,0}) \to Z^p(I^{\cdot,1}) \to \cdots$,

$B^p(C^\cdot) \to B^p(I^{\cdot,0}) \to B^p(I^{\cdot,1}) \to \cdots$, *and* $H^p(C^\cdot) \to H^p(I^{\cdot,0}) \to H^p(I^{\cdot,1}) \to \cdots$ *are also injective resolutions.*

Proof of the Grothendieck spectral sequence. Let A be an object of $\mathcal{A}$ and $0 \to A \to C^\cdot$ some injective resolution. If we we hit this with G, we get a complex $GC^\cdot$, of which we can find a fully injective resolution $I^{\cdot,\cdot}$, by the previous lemma. Then, note that $'E_1^{p,q} = H^q(FI^{p,\cdot}) = R^qF(GC^p)$, which is $(FG)C^p$ if $q = 0$ and 0 otherwise (using the fact that GC^p is F-acyclic). Therefore, $'E_2^{p,q}$ is $H^p((FG)C^\cdot) = R^p(FG)(A)$ for $q = 0$ and 0 otherwise. As a result, we have $''E_2^{p,q} \Rightarrow R^{p+q}(FG)(A)$, where we give the latter the obvious filtration (just itself and 0). So it remains to show that $''E_2^{p,q} = (R^pF)(R^qG)(A)$.

Note that $''E_1^{p,q} = H^q(FI^{\cdot,p})$, which one can check easily is just $FH^q(I^{\cdot,p})$, because we have a fully injective resolution. Then, $''E_2^{p,q} = H^p\left(FH^q(I^{\cdot,p})\right)$, and since $H^q(I^{\cdot,p})$ is an injective resolution of $H^q(GC^\cdot) = R^qG(A)$ by the definition of full injectivity, it follows that $''E_2^{p,q} = (R^pF)(R^qG)(A))$, as desired. ☺

As a corollary, we easily obtain the Leray spectral sequence.

Corollary 74 (Leray spectral sequence)**.** *Suppose* $f : X \to Y$ *is a continuous map of topological spaces. Then, let* $f_* : \mathsf{Sh}_X \to \mathsf{Sh}_Y$ *be the direct image functor from sheaves of abelian groups over* X *to sheaves of abelian groups over* Y*. Also, let* $\Gamma(X,-)$ *and* $\Gamma(Y,-)$ *be the global sections functor for* Sh_X *and* Sh_Y*, respectively. Then, for any sheaf* $\mathcal{F}$ *on* X*, there is a spectral sequence*

$$E_2^{p,q} = H^p(Y, R^qf_*\mathcal{F}) \Rightarrow H^{p+q}(X,\mathcal{F}).$$

Proof. Since f_* is right adjoint to f^{-1}, which is exact, lemma 61 tells us that f_* sends injectives to injectives, which are $\Gamma(Y,-)$-acyclic, so we can use the Grothendieck spectral sequence. The result follows. ☺

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