STAGE, Talk VII: Deligne's version of the Rankin method

Francesc Fité[1]

10/19/2020

[1]Many thanks to Bjorn Poonen for his invaluable help while preparing this talk. The lemma on slide 28 was explained by him to me.

Goal

This talk is based on the paper by Nick Katz:

- *A Note on Riemann Hypothesis for Curves and Hypersurfaces Over Finite Fields*, IMRN, Volume 2015, Issue 9, 2015, Pages 2328–2341.

The paper has two parts:

- The first explains how to reduce proving the Riemann hypothesis for *any* smooth, projective, geometrically connected curve over a finite field to proving the RH for the Fermat curves over finite fields. (Today)
- The second explains a reduction process that allows to prove the RH for *any* smooth and projective hypersurface by proving the RH for a *single* hypersurface in *each* degree, dimension, and characteristic. (Next week)

The zeta function

$\mathbb{F}_q$ denotes the finite field with $q = p^f$ elements, where p is a prime.

X smooth, projective, geometrically connected $\mathbb{F}_q$-variety.

Recall the zeta function of X

$$Z(X,T) = \exp\left(\sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n})\frac{T^n}{n}\right) \in \mathbb{Q}[[T]]\,.$$

For a closed point $\mathfrak{p} \in X$, let $k(\mathfrak{p})$ denote its residue field. Set

$$\mathrm{Nm}\,\mathfrak{p} := \#k(\mathfrak{p})\,, \qquad q^{\deg \mathfrak{p}} = \#k(\mathfrak{p})\,.$$

The zeta function can be rewritten as

$$Z(X,T) = \prod_{\text{closed } \mathfrak{p} \in X} (1 - T^{\deg \mathfrak{p}})^{-1} \in \mathbb{Q}[[T]]\,.$$

A cohomological expression

Let $d = \dim X$ and choose a prime $\ell \neq p$.

We saw that $Z(X, T)$ admits an ℓ-adic cohomological expression

$$Z(X,T) = \prod_{i=1}^{2d} P_i(X,T)^{(-1)^{i+1}}, \qquad \text{where } P_i(X,T) \in \mathbb{Q}_\ell[T].$$

This shows that $Z(X, T) \in \mathbb{Q}[[T]] \cap \mathbb{Q}_\ell(T) = \mathbb{Q}(T)$.

The polynomials $P_i(X, T)$ were described in the following way:

- Let $\sigma_q \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ be the arithmetic Frobenius, i.e, $\sigma_q(a) = a^q$.
- Let $F_q = \sigma_q^{-1}$ be the geometric Frobenius.
- Write $\overline{X} := X \times_{\mathrm{Spec}\, \mathbb{F}_q} \mathrm{Spec}\, \overline{\mathbb{F}}_q$.
- Then $P_i(X, T) = \det(1 - TF_q | H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell))$.
- In particular, $P_0(X, T) = 1 - T$ and $P_{2d}(X, T) = 1 - qT$.

The Riemann hypothesis for varieties over finite fields

$|\cdot|$ will denote the complex absolute value.

We say that $\alpha \in \overline{\mathbb{Q}}$ is a q^i-Weil number if $|\iota(\alpha)| = q^{i/2}$ for all $\iota\colon \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$.

Riemann hypothesis for X

For each i, the polynomial $P_i(X, T)$ belongs to $\mathbb{Q}[T]$ and its reciprocal roots are q^i-Weil numbers.

If $X = C$ is a curve, there is an explicit description of $H^1_{\text{ét}}(\overline{C}, \mathbb{Q}_\ell)$.

- Let g denote the genus of C.
- Let J denote the Jacobian of C.
- Define the ℓ-adic Tate module of J as

$$T_\ell(J) := \varprojlim_r J[\ell^r](\overline{\mathbb{F}}_q) \simeq \mathbb{Z}_\ell^{2g}\,.$$

The Riemann hypothesis for curves over finite fields

Define the rational ℓ-adic Tate module of J as

$$V_\ell(J) := T_\ell(J) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \,.$$

It turns out that

$$H^1_{\text{ét}}(\overline{C}, \mathbb{Q}_\ell) \simeq V_\ell(J)^\vee \qquad \text{as } \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)\text{-modules.}$$

We get

$$P_1(C,T) = \det(1 - T\sigma_q | V_\ell(J)) \,.$$

Note that

$$Z(C,T) = \frac{P_1(C,T)}{(1-T)(1-qT)} \in \mathbb{Q}(T) \quad \Rightarrow \quad P_1(C,T) \in \mathbb{Z}[T] \,.$$

Riemann hypothesis for C

The reciprocal roots of $P_1(C,T)$ are q-Weil numbers.

The étale fundamental group

Let:

- U be a connected scheme.
- $\xi : \operatorname{Spec} \Omega \to U$ be a geometric point of U (Ω a separably closed field).

In talk VI, we defined the étale fundamental group $\pi_1^{\text{ét}}(U, \xi)$, which classifies finite étale extensions of U:

$$\left\{ \begin{array}{c} \text{finite étale} \\ \text{covers of } U \end{array} \right\} \overset{\simeq}{\longleftrightarrow} \left\{ \begin{array}{c} \text{finite sets equipped with} \\ \text{a continuous action of } \pi_1^{\text{ét}}(U, \xi) \end{array} \right\}$$

Given a second geometric point ξ' of U, we have an isomorphism

$$\pi_1^{\text{ét}}(U, \xi) \simeq \pi_1^{\text{ét}}(U, \xi')$$

given by a “path" connecting ξ and ξ'. The isomorphism is independent of this path up to inner conjugacy.

Frobenius elements in $\pi_1^{\text{ét}}(U, \xi)$

Let k be a *finite* field.

The choice a separable closure k_s/k defines a geometric point κ of $\operatorname{Spec} k$ and we have

$$\pi_1^{\text{ét}}(\operatorname{Spec} k, \kappa) \simeq \operatorname{Gal}(k_s/k)\,.$$

Any $\operatorname{Spec} k \xrightarrow{u} U$ induces a map of fundamental groups

$$\Phi\colon \operatorname{Gal}(k_s/k) \to \pi_1^{\text{ét}}(U, \kappa) \simeq \pi_1^{\text{ét}}(U, \xi)$$

Let $F_k \in \operatorname{Gal}(k_s/k)$ denote the geometric Frobenius of k.

Define Frob_u as the image of F_k via Φ.

Frob_u is well-defined only as a *conjugacy class* of $\pi_1^{\text{ét}}(U, \xi)$.

Lisse $\overline{\mathbb{Q}}_\ell$-sheaves

From now on, assume that U is a connected $\mathbb{F}_q$-variety.

A lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on U of rank r is a continuous representation

$$\varrho\colon \pi_1^{\text{ét}}(U,\xi) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)\,.$$

One can show that $\mathrm{im}(\varrho) \subseteq \mathrm{GL}_r(E_\lambda)$, for some *finite* $E_\lambda/\mathbb{Q}_\ell$.

Let $\mathfrak{p}$ be a *closed* point of U. Then:

- The construction of "Frob_u" from the previous slide applies to $\mathfrak{p}$. Indeed, the residue field $k(\mathfrak{p})$ is *finite* and $\mathfrak{p}$ induces

  $$u_\mathfrak{p}\colon \mathrm{Spec}\, k(\mathfrak{p}) \to U\,.$$

  In this case, we will write $\mathrm{Frob}_\mathfrak{p} = \mathrm{Frob}_{u_\mathfrak{p}}$.
- Even though $\mathrm{Frob}_\mathfrak{p}$ is only well defined up to conjugacy, the polynomial $\det(1 - T\mathrm{Frob}_\mathfrak{p}|\mathcal{F})$ is uniquely defined.

L-function of a lisse $\overline{\mathbb{Q}}_\ell$-sheaf

Define the L-function of $\mathcal{F}$ on U as

$$L(U,\mathcal{F},T) := \prod_{\text{closed } \mathfrak{p}\in U} \det(1 - T^{\deg \mathfrak{p}}\mathrm{Frob}_\mathfrak{p}|\mathcal{F})^{-1} \in \overline{\mathbb{Q}}_\ell[[T]]\,.$$

Note that if $\mathcal{F}$ is trivial, then $L(U,\mathcal{F},T) = Z(U,T)$.

As for $Z(U,T)$, we would like to see that $L(U,\mathcal{F},T)$ admits an ℓ-adic cohomological expression.

In order to do so, we need three ingredients:

- Reinterpretation of “lisse sheaves on U" as “ℓ-adic sheaves on U".
- Cohomology with compact support “with coefficients": $H^i_{\text{ét},c}(\overline{U},\mathcal{F})$.
- A Grothendieck-Lefschetz trace formula “with coefficients".

ℓ-adic sheaves

Locally constant $\mathbb{Z}_\ell$-sheaf. It is a projective system $\mathcal{F} = (\mathcal{F}_r)_{r\geq 1}$ of abelian étale sheaves $\mathcal{F}_r$ satisfying:

- $\mathcal{F}_r$ has finite stalks.
- $\mathcal{F}_r$ is locally constant.
- $\mathcal{F}_r$ is killed by ℓ^r and $\mathcal{F}_{r+1}/\ell^r\mathcal{F}_{r+1} \simeq \mathcal{F}_r$.

Locally constant $\mathbb{Q}_\ell$-sheaf. It is a locally constant $\mathbb{Z}_\ell$-sheaf regarded up to isogeny (i.e., the Homs from $\mathcal{F}$ to $\mathcal{G}$ are $\mathrm{Hom}_{\mathbb{Z}_\ell\text{-sheaves}}(\mathcal{F},\mathcal{G}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$).

There is a correspondence:

$$\left\{ \begin{array}{c} \text{locally constant} \\ \mathbb{Q}_\ell\text{-sheaves on } U \end{array} \right\} \stackrel{\simeq}{\longleftrightarrow} \left\{ \begin{array}{c} \text{finite dim. continuous} \\ \mathbb{Q}_\ell\text{-representations of } \pi_1^{\text{ét}}(U,\xi) \end{array} \right\}$$

There are variants of the above obtained by replacing "$\mathbb{Q}_\ell$" with "E_λ".

Cohomology of ℓ-adic sheaves

One can define cohomology groups of an ℓ-adic sheaf $\mathcal{F} = (\mathcal{F}_r)_{r\geq 1}$ by

$$H^i_{\text{ét},c}(\overline{U}, \mathcal{F}) := \varprojlim_r H^i_{\text{ét},c}(\overline{U}, \mathcal{F}_r)\,.$$

Trivial example: If $\mathcal{F} := (\mathbb{Z}/\ell^r\mathbb{Z})_{r\geq 1}$, then $H^i_{\text{ét},c}(\overline{U}, \mathcal{F}) = H^i_{\text{ét},c}(\overline{U}, \mathbb{Z}_\ell)$.

If $d = \dim\, U$, then we have:

- $H^i_{\text{ét},c}(\overline{U}, \mathcal{F})$ is finite dimensional. It is 0 unless $i \in [0, 2d]$.
- If U is affine, then $H^i_{\text{ét},c}(\overline{U}, \mathcal{F}) = 0$ unless $i \in [d, 2d]$.
- Recall the exact sequence

$$0 \longrightarrow \underbrace{\pi_1^{\text{ét}}(\overline{U}, \zeta)}_{\pi_1^{\mathrm{geom}}} \longrightarrow \underbrace{\pi_1^{\text{ét}}(U, \zeta)}_{\pi_1^{\mathrm{arith}}} \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \longrightarrow 0\,.$$

$H^{2d}_{\text{ét},c}(\overline{U}, \mathcal{F})(d) =$largest quotient $\mathcal{F}_{\pi_1^{\mathrm{geom}}}$ of $\mathcal{F}$ on which π_1^{geom}acts trivially.

Generalized Grothendieck-Lefschetz trace formula

The correspondence:

$$\left\{ \begin{array}{c} \text{locally constant} \\ \mathbb{Q}_\ell\text{-sheaves on } U \end{array} \right\} \stackrel{\simeq}{\longleftrightarrow} \left\{ \begin{array}{c} \text{finite dim. continuous} \\ \mathbb{Q}_\ell\text{-representations of } \pi_1^{\text{ét}}(U,\xi) \end{array} \right\}$$

$$\mathcal{F} \longmapsto \varrho_{\mathcal{F}}$$

is such that, for every finite extension $k/\mathbb{F}_q$, the Grothendieck-Lefschetz trace formula (GLTF) is satisfied:

$$\sum_{u\in U(k)} \mathrm{Tr}(\varrho_{\mathcal{F}}(\mathrm{Frob}_u)) = \sum_{i=1}^{2d} (-1)^i \mathrm{Tr}\big(F_k | H^i_{\text{ét},c}(\overline{U},\mathcal{F})\big)\,.$$

More in general, the formula holds for lisse $\overline{\mathbb{Q}}_\ell$-sheaves (replace "$\mathbb{Q}_\ell$" by "$E_\lambda$").

Cohomological expression for the L-function

$$\sum_{u\in U(k)} \mathrm{Tr}(\varrho_{\mathcal{F}}(\mathrm{Frob}_u)) = \sum_{i=1}^{2d}(-1)^i\mathrm{Tr}\big(F_k|H^i_{\text{ét},c}(\overline{U},\mathcal{F})\big)\,.$$

Taking $k=\mathbb{F}_{q^n}$, $\mathcal{F}=(\mathbb{Z}/\ell^r\mathbb{Z})_{r\geq 1}$, we get the GLTF "without coefficients"

$$\#U(\mathbb{F}_{q^n}) = \sum_{i=1}^{2d}(-1)^i\mathrm{Tr}\big(F_q^n|H^i_{\text{ét},c}(\overline{U},\mathbb{Z}_\ell)\big)$$

As we did in Talk IV for $Z(U,T)$ using the GLTF "without coefficients", we can use the general GLTF to get a cohomological expression (CE)

$$L(U,\mathcal{F},T) = \prod_{i=1}^{2d}\det(1-TF_q|H^i_{\text{ét},c}(\overline{U},\mathcal{F}))^{(-1)^{i+1}} \in \overline{\mathbb{Q}}_\ell(T)\,.$$

Case of interest

We will consider the setting in which:

- $U := U_0$ is a smooth, *affine*, geometrically conn. *curve* over $\mathbb{F}_q$.
- $f\colon \mathcal{X} \to U_0$ is a proper and smooth morphism.
- $\mathcal{F} := R^i f_* \mathbb{Q}_\ell$.

This might look surprising since our original interest was to study: $H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ for some smooth, projective, geom. connected $\mathbb{F}_q$-variety X.

What motivates the above setting is the following strategy:

- Put X in the family $\mathcal{X}$, i.e. obtain X as a fibre of $f\colon \mathcal{X} \to U_0$.
- Via proper base change, reduce:

$$\text{study of } H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \qquad \text{to} \qquad \text{study of } H^j_{\text{ét},c}(\overline{U}_0, R^i f_* \mathbb{Q}_\ell).$$

Moral: At the cost of complicating the “sheaf of coefficients" we greatly simplify the geometry of the “scheme".

Proper base change

Let $\mathfrak{p}$ be a closed point of U_0. Define $X_\mathfrak{p}$, $\overline{X}_\mathfrak{p}$ via the cartesian diagram:

$$\begin{array}{ccccc} \overline{X}_\mathfrak{p} & \longrightarrow & X_\mathfrak{p} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow f \\ \operatorname{Spec} \overline{k(\mathfrak{p})} & \longrightarrow & \operatorname{Spec} k(\mathfrak{p}) & \xrightarrow{u_\mathfrak{p}} & U_0 \end{array}$$

Proper base change (PBC) provides an isomorphism

$$(u_\mathfrak{p})^*(R^i f_* \mathbb{Q}_\ell) \simeq H^i_{\text{ét}}(\overline{X}_\mathfrak{p}, \mathbb{Q}_\ell)$$

of $\mathbb{Q}_\ell$-adic sheaves on $\operatorname{Spec} k(\mathfrak{p})$. This means:

$$\det(1 - T\mathrm{Frob}_\mathfrak{p} | R^i f_* \mathbb{Q}_\ell) = \det(1 - T F_q^{\deg\,\mathfrak{p}} | H^i_{\text{ét}}(\overline{X}_\mathfrak{p}, \mathbb{Q}_\ell)) \,.$$

(Recall $\# k(\mathfrak{p}) = q^{\deg \mathfrak{p}}$).

Recap

We put together what we have seen so far:

We are given a proper smooth family $f\colon \mathcal{X} \to U_0$ over an affine curve.

Then, on the one hand, for $\mathcal{F} = R^i f_* \mathbb{Q}_\ell$, we have seen

$$\begin{aligned} L(U_0, R^i f_* \mathbb{Q}_\ell, T) &=^{def} \prod_{\text{closed } \mathfrak{p} \in U_0} \det(1 - T^{\deg \mathfrak{p}} \mathrm{Frob}_{\mathfrak{p}} | R^i f_* \mathbb{Q}_\ell)^{-1} \\ &=^{PBC} \prod_{\text{closed } \mathfrak{p} \in U_0} \det(1 - (TF_q)^{\deg \mathfrak{p}} | H^i_{\text{ét}}(\overline{X}_{\mathfrak{p}}, \mathbb{Q}_\ell))^{-1}. \end{aligned}$$

On the other hand, for a general lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$, we have seen

$$\begin{aligned} L(U_0, \mathcal{F}, T) &=^{CE} \det(1 - TF_q | H^1_{\text{ét},c}(\overline{U}_0, \mathcal{F})) \cdot \det(1 - TF_q | H^2_{\text{ét},c}(\overline{U}_0, \mathcal{F}))^{-1} \\ &=^{H^2} \det(1 - TF_q | H^1_{\text{ét},c}(\overline{U}_0, \mathcal{F})) \cdot \det(1 - qTF_q | \mathcal{F}_{\pi_1^{\mathrm{geom}}})^{-1}. \end{aligned}$$

Some types of lisse sheaves

Let $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $U_0/\mathbb{F}_q$ and let $\iota\colon \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$.

- $\mathcal{F}$ is said to be ι-pure of weight $w \in \mathbb{Z}$ if, for every closed point $\mathfrak{p} \in U_0$ and every eigenvalue α of $\mathrm{Frob}_\mathfrak{p}|\mathcal{F}$, we have

$$|\iota(\alpha)| = (\mathrm{Nm}\,\mathfrak{p})^{w/2}\,.$$

- $\mathcal{F}$ is pure of weight w if it is ι-pure of weight w for every choice of ι.
- $\mathcal{F}$ is ι-real (resp. ι-integral) if, for every closed $\mathfrak{p} \in U_0$, the polynomial $\det(1 - T\mathrm{Frob}_\mathfrak{p}|\mathcal{F})$ has real (resp. integral) coefficients.

Remarks:

- RH holds for the $X_\mathfrak{p}$'s $\Longleftrightarrow$ $R^i f_* \mathbb{Q}_\ell$ is pure for all i.
- RH holds for the $X_\mathfrak{p}$'s $\Longrightarrow P_i(X_p, T) \in \mathbb{Z}[T] \Longrightarrow R^i f_* \mathbb{Q}_\ell$ is integral.
- If $\mathcal{X} = \mathcal{C}/U_0$ is a curve, then:

$$Z(C_p, T) \in \mathbb{Q}(T) \Rightarrow P_1(C_p, T) \in \mathbb{Z}[T] \Rightarrow R^1 f_* \mathbb{Q}_\ell \text{ is integral.}$$

Deligne's version of the Rankin method (after Katz)

Theorem

Let $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U_0 such that:

- It is ι-real.
- For all $k \in \mathbb{Z}_{\geq 1}$, every eigenval. β of $F_q|(\mathcal{F}^{\otimes 2k})_{\pi_1^{\mathrm{geom}}}$ has $|\iota(\beta)| \leq 1$.

Then, for all closed points $\mathfrak{p}$, every eigenval. α of $\mathrm{Frob}_\mathfrak{p}|\mathcal{F}$ has $|\iota(\alpha)| \leq 1$.

Proof: Via ι, we regard $\overline{\mathbb{Q}}_\ell \subseteq \mathbb{C}$.

$$L_{\mathfrak{p},2k}(T) := \frac{1}{\det(1 - T^{\deg \mathfrak{p}}\mathrm{Frob}_\mathfrak{p}|\mathcal{F}^{\otimes 2k})} = \exp\left(\sum_{n\geq 1} \mathrm{Tr}(\mathrm{Frob}_\mathfrak{p}^n|\mathcal{F})^{2k}\frac{T^{n\deg(\mathfrak{p})}}{n}\right).$$

$\mathcal{F}$ is ι-real $\Longrightarrow \mathrm{Tr}(\mathrm{Frob}_\mathfrak{p}^n|\mathcal{F})^{2k} \in \mathbb{R}_{\geq 0} \Longrightarrow L_{\mathfrak{p},2k}(T) \in \mathbb{R}_{\geq 0}[[T]]$.

Since $L(U_0, \mathcal{F}^{\otimes 2k}, T) = \prod_\mathfrak{p} L_{\mathfrak{p},2k}(T)$, we have that $L(U_0, \mathcal{F}^{\otimes 2k}, T)$ dominates, coefficient by coefficient, each of its Euler factors $L_{\mathfrak{p},2k}(T)$.

If $\mathrm{R}_{\mathrm{conv}}$:="Radius of convergence", then

$$\mathrm{R}_{\mathrm{conv}}\left(L(U_0,\mathcal{F}^{\otimes 2k},T)\right)\leq \mathrm{R}_{\mathrm{conv}}\left(L_{\mathfrak{p},2k}(T)\right)\,,$$

that is

$$\mathrm{R}_{\mathrm{conv}}\left(\frac{\det(1-TF_q|H^1_{\text{ét},c}(\overline{U}_0,\mathcal{F}^{\otimes 2k}))}{\det(1-qTF_q|(\mathcal{F}^{\otimes 2k})_{\pi_1^{\mathrm{geom}}})}\right)\leq \mathrm{R}_{\mathrm{conv}}\left(\frac{1}{\det(1-T^{\deg\mathfrak{p}}\mathrm{Frob}_{\mathfrak{p}}|\mathcal{F}^{\otimes 2k})}\right)\,,$$

that is

$$\min|\text{roots of LHS den.}|\leq \min|\text{roots of RHS den.}|\,,$$

that is

$$\max_{\alpha \text{ eigenval. of } \mathrm{Frob}_{\mathfrak{p}}|\mathcal{F}}|\alpha|^{2k/\deg\mathfrak{p}}\leq \max_{\beta \text{ eigenval. of } F_q|(\mathcal{F}^{\otimes 2k})_{\pi_1^{\mathrm{geom}}}} q\cdot|\beta|\leq q.$$

Therefore

$$|\alpha|\leq q^{\deg\mathfrak{p}/(2k)}\longrightarrow 1\qquad \text{as } k\to\infty\,.$$

Q.E.D.

Corollary (Katz)

Let $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U_0 such that:

- *It is ι-real.*
- *There exists a closed point $\mathfrak{p}_0$ such that every eigenval. $\alpha_{0,i}$ of $\mathrm{Frob}_{\mathfrak{p}_0}|\mathcal{F}$ has $|\iota(\alpha_{0,i})| \leq 1$.*

Then, for all closed points $\mathfrak{p}$, every eigenval. α of $\mathrm{Frob}_{\mathfrak{p}}|\mathcal{F}$ has $|\iota(\alpha)| \leq 1$.

Proof: Again, regard $\overline{\mathbb{Q}}_\ell \subseteq \mathbb{C}$ via ι.

By the theorem it suffices to show that

for all $k \in \mathbb{Z}_{\geq 1}$, every eigenvalue β of $F_q|(\mathcal{F}^{\otimes 2k})_{\pi_1^{\mathrm{geom}}}$ has $|\beta| \leq 1$.

Let $d := \deg \mathfrak{p}_0$. Then F_q^d acts on $(\mathcal{F}^{\otimes 2k})_{\pi_1^{\mathrm{geom}}}$ as $\mathrm{Frob}_{\mathfrak{p}_0}$.

Thus:

$$\beta^d = \text{some eigenvalue of } \mathrm{Frob}_{\mathfrak{p}_0}|(\mathcal{F}^{\otimes 2k})_{\pi_1^{\mathrm{geom}}} = \alpha_{0,i_1} \ldots \alpha_{0,i_{2k}}\,.$$

The hypothesis $|\alpha_{0,i}| \leq 1$ implies $|\beta| \leq 1$.

Q.E.D.

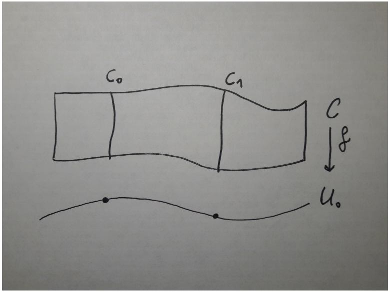
The “connect by curves" lemma

- nice= “smooth, projective, and geometrically connected".
- nice-affine=“smooth, *affine*, and geometrically connected".

Lemma (Katz)

Let C_0, C_1 be nice curves defined over $\mathbb{F}_q$ of genus $g \geq 1$.

After passing to a finite extension $E/\mathbb{F}_q$, C_0 and C_1 are fibres of a smooth and proper $f\colon \mathcal{C} \to U_0$ to a nice-affine curve U_0.



Idea of proof: For $g = 1$, after finite base change, both curves become elliptic curves equipped with a rational point of order $N \geq 4$. Take

$$\begin{array}{ll} \mathcal{C} := \mathcal{Y}_1(N) & \text{universal modular curve of level } \Gamma_1(N) \\ \Big\downarrow f & \\ U_0 := Y_1(N) & \text{affine modular curve of level } \Gamma_1(N) \end{array}$$

For $g \geq 2$, Deligne and Mumford provide a smooth, quasi-projective, and geom. connected scheme $H_g^0/\mathbb{F}_p$ classifying 3-canonically embedded genus g curves over $\mathbb{F}_p$. Take f defined by the cartesian diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{H}_g^0 \\ \Big\downarrow f & & \Big\downarrow \\ U_0 & \xrightarrow{\pi} & H_g^0 \end{array}$$

where U_0 is a nice-affine curve and π is bijective on $\mathbb{F}_q$-points (See: N. Katz, *Space filling curves over finite fields*, MRL, 1999).

Corollary (Katz)

Let g be an integer ≥ 1 *and p a prime.*

Suppose that there exists a nice genus g curve $C_0/\mathbb{F}_{p^{n_0}}$*, for some* $n_0 \geq 1$*, such that the RH holds for* C_0*.*

Then the RH holds for any nice genus g curve $C_1/\mathbb{F}_{p^{n_1}}$*, for any* $n_1 \geq 1$*.*

Proof: Since the RH is insensitive to finite base change, we may assume that C_0 and C_1 are both defined $\mathbb{F}_q$.

View C_0 and C_1 as fibres of a family $f : \mathcal{C} \to U_0$ as in the Lemma.

By choosing a square root $q^{1/2}$ of q in $\overline{\mathbb{Q}}_\ell$, on can define a lisse sheaf

$$\mathcal{F} := R^1 f_* \mathbb{Q}_\ell(1/2)$$

The eigenvalues of $\mathrm{Frob}_\mathfrak{p}|\mathcal{F}$ are those of $R^1 f_* \mathbb{Q}_\ell$ divided by $(\mathrm{Nm}\,\mathfrak{p})^{1/2}$.

(Via this normalization, RH is the assertion that $\mathcal{F}$ is pure of weight 0).

Let ι be any $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$, and regard $\mathbb{Q}_\ell \subseteq \mathbb{C}$ via ι.
Because RH holds for C_0, every eigenvalue γ_0 of $\mathrm{Frob}_{u_0}|\mathcal{F}$ has

$$|\gamma_0| \leq 1\,.$$

Since $\mathcal{F}$ is ι-real, this implies that every eigenvalue γ_1 of $\mathrm{Frob}_{u_1}|\mathcal{F}$ has

$$|\gamma_1| \leq 1\,.$$

Thus, every inverse root α_1 of $P_1(C_1, T) = \det(1 - TF_q|H^1_{\text{ét}}(\overline{C}_1, \mathbb{Q}_\ell))$ has

$$|\alpha_1| \leq q^{1/2}.$$

Because of the functional equation satisfied by $P_1(C_1, T)$, the map

$$\alpha_1 \mapsto q/\alpha_1$$

is an involution of the roots of $P_1(C_1, T)$. Thus

$$|\alpha_1| \geq q^{1/2}.$$

Q.E.D.

Fermat curves and Jacobi sums

Fermat curves. Let d be an integer ≥ 2 and coprime to p. The Fermat curve F_d of degree d is the curve in $\mathbb{P}^2_{F_q}$ determined by the equation

$$u^d + v^d = w^d \,.$$

Jacobi sums. Given multiplicative characters $\chi_1, \chi_2 \colon \mathbb{F}_q^\times \to \mathbb{C}^\times$, define

$$J(\chi_1, \chi_2) := \sum_{a \in \mathbb{F}_q} \chi_1(a)\chi_2(1-a) \,.$$

(With the convention that $\chi_i(0) = 0$). Basic properties:

- $J(1,1) = q-2$ and $J(1,\chi) = -1$ for χ nontrivial.
- $J(\chi, \chi^{-1}) = -\chi(-1)$ for χ nontrivial.
- If χ_1, χ_2 and $\chi_1 \cdot \chi_2$ are nontrivial, then $|J(\chi_1, \chi_2)| = q^{1/2}$.

RH for Fermat curves

Theorem (Weil)

Let $e = \gcd(d, q-1)$ *and* $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ *a character of order e. Then*

$$P_1(F_d, T) = \prod_{\substack{a,b=1 \\ a+b\neq e}}^{e-1} (1 + J(\chi^a, \chi^b)T)$$

1) Using that $\#\{x \in \mathbb{F}_q \mid x^e = t\} = 1 + \sum_{a=1}^{\ell-1} \chi^a(t)$, one shows

$$\#F_d(\mathbb{F}_{q^n}) = 1 + q^n - \sum_{\substack{a,b=1 \\ a+b\neq e}}^{e-1} (-J(\chi^a, \chi^b))^n$$

2) Use the fact:

$$\left.\begin{array}{c} \exp\left(\sum_{n=1}^\infty \left(\sum_j \beta_j^n - \sum_j \alpha_j^n\right)\frac{T^n}{n}\right) = \frac{P(T)}{Q(T)} \\ P(T), Q(T) \in \mathbb{Q}[T], P(0) = Q(0) = 1 \end{array}\right\} \Longrightarrow \begin{array}{c} P(T) = \prod_j (1 - \alpha_j T) \\ Q(T) = \prod_j (1 - \beta_j T) \end{array}$$

Completion of the proof of RH for nice curves

Knowing RH for Fermat curves, does not imply (immediately) RH for every nice curve, since $\mathrm{genus}(F_d) = \binom{d-1}{2}$, for example.

Lemma

For every $g \geq 1$ and prime p, there exists $d \geq 2$ coprime to p such that the Fermat curve F_d over $\mathbb{F}_p$ has a nice genus g quotient C.

Remark: This conclude s the proof of RH for nice curves, since

$$V_\ell(J_C) \subseteq V_\ell(J_{F_d}) \Rightarrow P_1(C,T) \mid P_1(F_d,T) \Rightarrow \text{RH holds for C}\,.$$

Proof of the Lemma: If $p \neq 2$, let $d' = 2g+1$ or $2g+2$ so that $p \nmid d'$. Then $F_{2d'}$ has an obvious quotient map to $x^{d'} + y^2 = 1$.

If $p = 2$, note that F_{2g+1} has a quotient map to $y^2 + y + x^{2g+1} = 0$.

Corollary

The RH holds for any nice curve defined over a finite field.

Persistence of purity

Theorem (Katz)

Let $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U_0 such that:

- *It is ι-real.*
- *There exists a closed point $\mathfrak{p}_0$ such that every eigenval. $\alpha_{0,i}$ of $\mathrm{Frob}_{\mathfrak{p}_0}|\mathcal{F}$ has $|\iota(\alpha_{0,i})| = 1$.*

Then, $\mathcal{F}$ is ι-pure of weight 0.

Proof: Applying the corollary we aim to refine, we have:

for all closed points $\mathfrak{p}$, every eigenvalue α of $\mathrm{Frob}_{\mathfrak{p}}|\mathcal{F}$ has $|\alpha| \leq 1$.

Therefore it suffices to show that $\det \mathcal{F}$ is ι-pure of weight 0. We may thus assume that $\mathcal{F}$ has rank 1. The theorem then follows from:

Lemma

Let $\mathcal{L}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U_0 of rank 1. Then there exist an integer n and $\alpha \in \overline{\mathbb{Q}}_\ell^\times$ such that $\mathrm{Frob}_{\mathfrak{p}}|\mathcal{L}^{\otimes n} = \alpha^{\deg \mathfrak{p}}$ for every closed point $\mathfrak{p}$.

Proof of the Lemma:

Since RH holds for the complete nonsingular model of U_0:

every eigenvalue α of $F_q|H^1_{\text{ét},c}(\overline{U}_0, \overline{\mathbb{Q}}_\ell)$ has $|\alpha| \leq q^{1/2}$.

By Poincaré duality:

every eigenval. α of $F_q|H^1_{\text{ét}}(\overline{U}_0, \overline{\mathbb{Q}}_\ell)$ has $|\alpha| \geq q^{1/2}$. In particular $\alpha \neq 1$.

Recall that $\mathcal{L}$ is a continuous homomorphism

$$\mathcal{L} \colon \pi_1^{\text{arith}} \to \mathcal{O}_{E_\lambda}^\times \subseteq \overline{\mathbb{Q}}_\ell^\times .$$

By replacing $\mathcal{L}$ with $\mathcal{L}^{\otimes \ell \cdot \# \mathbb{F}_\lambda}$, we may assume that

$$\mathcal{L} \colon \pi_1^{\text{arith}} \to 1 + \ell\lambda\mathcal{O}_{E_\lambda} \simeq \ell\lambda\mathcal{O}_{E_\lambda} \subseteq \overline{\mathbb{Q}}_\ell .$$

Then $\mathcal{L}|_{\pi_1^{\text{geom}}} \in H^1_{\text{ét}}(\overline{U}_0, \overline{\mathbb{Q}}_\ell)$ fixed by F_q, so $\mathcal{L}|_{\pi_1^{\text{geom}}}$ must be trivial.

Q.E.D.