PERFECTOID RINGS VIA PERFECT PRISMS

The themes of this talk are: for *perfect* $\mathbf{F}_p$-algebras, derived and classical constructions often coincide. This induces similar behaviour on perfect prisms. These completely characterize perfectoid rings, which leads to similar "coincidences" for perfectoid rings as well.

We begin with a brief recollection of a definition and some lemmas from previous talks.

Definition 1. Let (A, I) be a pair with A a δ-ring and $I \subseteq A$ an ideal. Then (A, I) is called a prism if $I \subseteq A$ is a Cartier divisor such that A is derived (p, I)-complete and $p \in (I, \phi(I))$. A prism is called perfect if ϕ is an isomorphism.

Here is a compilation of some results from previous talks.

Lemma 2. *The following statements are true:*

(a) *The category of perfect $\mathbf{F}_p$-algebras is equivalent to the category of p-complete perfect δ-rings via the Witt vector construction.*

(b) *If (A, I) is a perfect prism, then $I = (d)$ for some distinguished element $d \in A$ (i.e., $\delta(d) \in A^\times$).*

(c) *An element $\sum_{i \geq 0} [a_i] p^i \in A$ is distinguished iff $a_1 \in A/p$ is a unit.*

Let us now turn to the content of this talk; we will define the terms involved below in due course.

Theorem 3. *The category of perfectoid rings is equivalent to the category of perfect prisms; this equivalence sends a perfect prism (A, I) to A/I.*

If the reader has not seen the definition of perfectoid rings, they can take the above theorem as a definition, and regard the remainder of this talk as establishing good properties of this class of rings. Bhatt's notes adopt this approach; however, since I suspect many in the audience have seen the more "conventional" definition of perfectoid rings, this approach may cause some psychological confusion during the talk. We will therefore make the following definition:

Definition 4. A *perfectoid'* ring is a commutative ring which can be written as a quotient A/I for some perfect prism (A, I).

Then, Theorem 3 states that perfectoid' = perfectoid. First, we recall a simple lemma.

Lemma 5. *Let B be a perfect $\mathbf{F}_p$-algebra, and let S be a classically p-complete ring. Then every map $B \to S/p$ admits a lift $W(B) \to S$.*

Proof sketch. The proof is via obstruction theory: it suffices to define successive lifts $W(B)/p^n \to S/p^n$ for each $n \geq 1$. The obstruction to doing so lies in some Ext-group involving the cotangent complex $L_{B/\mathbf{F}_p}$. However, $L_{B/\mathbf{F}_p} = 0$ (so all such obstructions vanish). To see this, observe that the Frobenius on B induces an *automorphism* of $L_{B/\mathbf{F}_p}$; however, it is also the zero morphism by the classical calculation $d(x^p) = 0$. $\square$

Corollary 6. *Let R be a p-complete ring, and let $R^\flat = \varprojlim_\phi R/p$ be the limit perfection of R/p. Then the map $R^\flat \to R/p$ lifts (by Lemma 5) to a map $\theta : W(R^\flat) \to R$.*

Now we show:

Proposition 7. *The category of perfectoid' rings is equivalent to the category of perfect prisms.*

Proof. A perfect prism (A, I) is sent to A/I. The hard part is showing that this functor is essentially surjective. Assume R is a perfectoid' ring, so $R = A/I$ for some perfect prism (A, I). We will describe how to reconstruct A and I.

(a) We begin by reconstructing A given R. In fact, $A \cong W(R^\flat)$ as p-complete δ-rings. To see this, it suffices to show (by Lemma 2(a)) that $A/p \simeq R^\flat$. Now $R/p = A/(p, I)$, so $R^\flat$ is $\varprojlim_\phi A/(p, I)$. By Lemma 2(b), the ideal I is generated by a distinguished $d \in A$, so the Frobenius on $A/(p, I)$ can be identified with the map $A/(p, d^p) \to A/(p, d)$. Taking the inverse limit, we see that $R^\flat$ is isomorphic to the classical d-adic completion $(A/p)^\wedge_d$. It therefore remains to see that A/p is d-adically complete. We will prove this in Lemma 8 below.

(b) We now reconstruct I. For this, we construct a surjection $A \to R$, i.e., a surjection $W(R^\flat) \to R$. Let $R^\flat \to R/p$ denote the evident surjection. Because $R^\flat$ is perfect, we can use Lemma 5 to see that this lifts to a map $\theta : W(R^\flat) \to R$. This map is still surjective, so $R = W(R^\flat)/\ker(\theta)$. We may therefore take $I = \ker(\theta)$.

□

We owe the reader the following:

Lemma 8. *Let R be a* perfect *$\mathbf{F}_p$-algebra, and let $x \in R$. Then the derived and classical x-adic completions of R coincide. In fact, R has bounded x-power torsion.*

Proof. The first claim follows from the second. Suppose $y \in R$ is x-power torsion, so $x^{p^N} y = 0$ for some $N \gg 0$. Then $x^{p^N} y^{p^{Nd}} = 0$ for $d \geq 0$. Because R is perfect, we can take p^{Nd}th roots to get $x^{1/p^d} y = 0$. This implies that $R[x^\infty] = R[x^{1/p^d}]$ for all $d \geq 0$. This finishes the proof, since $R[x^\infty] = R[x^{1/p^\infty}]$. □

Remark 9. The experienced arithmagician may recognize that the pair $(W(R^\flat), \ker(\theta))$ appearing in the proof of Proposition 7 is just the classical $\mathbf{A}_{\mathrm{inf}}$-construction. In fact, if R is a p-complete ring, one defines $\mathbf{A}_{\mathrm{inf}}(R) = W(R^\flat)$, so that there is a map $W(R^\flat) \to R$ as in Corollary 6.

One of the goals of this talk is to prove the following.

Theorem 10. *A commutative ring R is perfectoid' if and only if the following hold:*

(a) *R is (classically) p-complete.*
(b) *The Frobenius on R/p is surjective.*
(c) *The kernel of the map $\theta : \mathbf{A}_{\mathrm{inf}}(R) \to R$ is principal.*
(d) *There is $\pi \in R$ such that $\pi^p = pu$ for some unit $u \in R$.*

Moreover, if R is p-torsionfree, then condition (c) may be replaced by:

(c') *If $x \in R[1/p]$ and $x^p \in R$, then $x \in R$.*

This is just a restatement of Theorem 3, since the conditions (a)–(d) are the *definition* of a perfectoid ring.

Proof. Let us first show that a perfectoid' ring R satisfies conditions (a)–(d). Let $A = \mathbf{A}_{\mathrm{inf}}(R)$, and let $I = \ker(\theta)$, so $I = (d)$ for some distinguished $d \in A$.

(a) To show that R is classically p-complete, recall that $R = A/I$, and A is *derived* (p, I)-complete. Using that A is perfect, we can show that it is in fact *classically* (p, I)-complete (which implies that R is classically p-complete). Indeed, A/p is perfect and derived I-complete. This implies by Lemma 8 that A/p is classically I-complete. Inducting up, one sees that A/p^n is classically I-complete for all $n \geq 1$. But A is p-torsionfree, so since

$A = \varprojlim A/p^n$ by derived p-completeness, we see that A is classically (p, I)-complete, as desired.

(b) Note that $R/p = A/(p, d)$. Therefore, $A/p = R^\flat$ is perfect, which implies that the Frobenius on R/p is surjective.

(c) The kernel of θ is the ideal I, which we know is generated by a distinguished element $d \in A$.

(d) Let $\sum_{i \geq 0} [a_i] p^i$ be the Teichmüller expansion of d, so that $a_1 \in R^\flat$ is a unit. Thus $u := -\sum_{i \geq 1} [a_i] p^{i-1}$ is a unit in A, so that $d = [a_0] - pu$. Let $\pi \in R$ denote the image of $[a_0^{1/p}]$ in R, so that $\pi^p = pu$.

Although not yet relevant, let us make the following important observation. Since the Frobenius $R/p \to R/p$ can be identified with the Frobenius $A/(p, \pi^p) \to A/(p, \pi^p)$, we see that the kernel is generated by π since A is perfect. In other words, π generates the kernel of the Frobenius $R/p \to R/p$.

Conversely, let R be a ring satisfying conditions (a)–(d) (i.e., a perfectoid ring in the classical sense). We will show that R is perfectoid', with associated perfect prism $(A, I) = (\mathbf{A}_{\mathrm{inf}}(R), \ker(\theta))$. Since A is a perfect δ-ring, we only need to show that θ is surjective and that it is generated by a distinguished element. To see that θ is surjective, note that (by p-completeness) it suffices to prove that $R^\flat \to R/p$ is surjective. But this follows from assumption (b). We know from assumption (c) that $\ker(\theta)$ is principal, so let d be a generator. We need to show that d is distinguished.

For this, we first observe that A is (p, d)-complete. Indeed, it suffices to show that $A/p = R^\flat$ is d-complete, but this is clear since $R^\flat$ is y-complete for any $y \in \ker(R^\flat \to R/p)$. Let $\pi, u \in R$ be such that $\pi^p = pu$, and lift these to $x, v \in A$. Since u is a unit in R, we see that v is a unit in A. Similarly, x is in the Jacobson radical of A. Let $g = pv - x^p \in A$, so that $\theta(g) = pu - \pi^p = 0$, i.e., $g \in \ker(\theta)$. In particular, g is a multiple of d. To show that d is distinguished, it suffices (by the "irreducibility" of distinguished elements) to show that g is distinguished. To see this, note that by Lemma 2(c), it suffices to show that the coefficient of p in the Teichmüller expansion of g is invertible. But this coefficient is $[\overline{v}]$, where $\overline{v} = v \pmod p \in R^\flat$; this is a unit since v is a unit in A.

Let us now move on to the p-torsionfree case. Let us first show that if R is a p-torsionfree perfectoid' ring, then R satisfies (c'); so let $x \in R[1/p]$ be such that $x^p \in R$. Let π be as in condition (d) (we know by the above discussion that such a π exists). Then R is π-torsionfree (since it is p-torsionfree). Let n be the smallest nonnegative integer such that $y := \pi^n x \in R$; we need to show that $n = 0$, so assume for the sake of contradiction that $n > 0$. Then

$$y^p = (\pi^n x)^p = \pi^{np} x^p \subseteq (\pi^p),$$

so the image of y^p under the quotient $R \to R/\pi^p$ is zero. Since the Frobenius $R/\pi \to R/\pi^p$ is bijective (owing to the fact that π generates the kernel of the Frobenius on R/p), we see that the image of y under the quotient $R \to R/\pi$ is zero, i.e., $y \in (\pi)$. Since R is π-torsionfree, we see that $\pi^{n-1} x \in R$, which contradicts the minimality of n.

We now need to show that if R is a p-torsionfree perfectoid ring which satisfies (a), (b), (c'), and (d), then R satisfies (c). In other words, we need to show that the kernel of $\theta : W(R^\flat) \to R$ is principal. It suffices to show that the kernel of $R^\flat \to R/p$ is principal, since lifting any generator of this kernel to $\ker(\theta)$ defines a generator. In turn, it suffices to show the following:

($*$) The kernel of the Frobenius $\phi : R/p \to R/p$ is principally generated by $\pi \in R$ such that $\pi^p = pu$ for some unit $u \in R$. In particular, the Frobenius factors as

$$R/p \to R/\pi \xrightarrow{x \mapsto x^p} R/p.$$

Indeed, since the Frobenius on R/p is surjective by (b), we can define $\pi^\flat \in R^\flat$ via a choice of a compatible system $\{\overline{\pi}^{1/p^n}\}$ of pth power roots of $\overline{\pi} = \pi \pmod p \in R/p$. The claim is then that $\pi^\flat$ generates $\ker(R^\flat \to R/p)$. This follows if we show that $\pi^{1/p^{n-1}}$ generates $\ker(\phi^n : R/p \to R/p)$. By induction on n (with base case given by $(*)$), it follows that

$$\ker(\phi^n) = \phi^{-1}\ker(\phi^{n-1}) = \phi^{-1}(\pi^{1/p^{n-2}}) = (\pi^{1/p^{n-1}}).$$

The final equality is the factorization of ϕ as in $(*)$.

We now prove $(*)$. Suppose $x \in R$ is such that $x^p \in pR$ (so $x \pmod p$ is in the kernel of $\phi : R/p \to R/p$). Then $x^p = \pi^p y$ for some $y \in R$ since $\pi^p = pu$. Therefore the pth power of $\frac{x}{\pi} \in R[1/p]$ is in R, and so $\frac{x}{\pi} \in R$ (by (c')). In other words, $x \in (\pi)$, as desired. This immediately implies the desired factorization of $\phi : R/p \to R/p$. □

Let us now use Theorem 3 to prove some properties of perfectoid rings. We will first state the main results, as well as one consequence, and then give their proofs.

Theorem 11. *Let A, B, and C be perfectoid rings, and let $B \otimes_A^L C$ denote the derived pushout (in the category of rings) in the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & B \otimes_A^L C, \end{tikzcd}$$

and let $B\widehat{\otimes}_A^L C$ denote its derived p-completion. Then $B\widehat{\otimes}_A^L C$ is concentrated in degree zero, and it is a perfectoid ring.

Theorem 12. *Let R be a perfectoid ring, and let $\overline{R} = R/\sqrt{pR}$, $S = R/R[\sqrt{pR}]$, and $\overline{S} = S/\sqrt{pS}$. Then $\overline{R}$, S, and $\overline{S}$ are perfectoid, and the square*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \overline{R} \arrow[d] \\ S \arrow[r] & \overline{S} \end{tikzcd}$$

is a pullback and pushout square. Moreover:

(a) *S is p-torsionfree.*
(b) *$\sqrt{pR}$ is mapped isomorphically to $\sqrt{pS}$.*
(c) *$R[\sqrt{pR}]$ maps isomorphically onto $\ker(\overline{R} \to \overline{S})$.*

Corollary 13. *A perfectoid ring R is reduced.*

Proof. In the setup of Theorem 12, we know that S is p-torsionfree, and $\overline{R}$ and $\overline{S}$ are perfect $\mathbf{F}_p$-algebras. Therefore, we may assume R is either p-torsionfree or a perfect $\mathbf{F}_p$-algebra. Since perfect $\mathbf{F}_p$-algebras are reduced, we may assume R is p-torsionfree. If $\pi \in R$ is such that $\pi^p = pu$ for some unit $u \in R$, and $x \in R$ is such that $x^p = 0$, then we will show by induction that $x \in (\pi^n)$ for all n. This is clearly true for $n = 0$, establishing the base case. So assume $x = \pi^n y$ for some $y \in R$. Then $x^p = \pi^{np} y^p = 0$, so by π-torsionfreeness, we see that $y^p = 0$. But then $y \pmod p$ is in the kernel of $\phi : R/p \to R/p$. Since this kernel is generated by π, we have $y \in (\pi)$, so $x \in (\pi^{n+1})$. □

Let us now prove Theorem 11.

Proof of Theorem 11. Let $R = B^\flat \otimes_{A^\flat} C^\flat$. Then, we claim that $W(R)$ is the p-completed derived pushout $W(B^\flat)\widehat{\otimes}^L_{W(A^\flat)} W(C^\flat)$. It suffices to prove this mod p, i.e., that R the *derived* pushout $B^\flat \otimes^L_{A^\flat} C^\flat$. In fact, a stronger result is true:

($*$) derived and classical tensor products of *perfect* $\mathbf{F}_p$-algebras agree. In other words, $\mathrm{Tor}_n^{A^\flat}(B^\flat, C^\flat) = 0$ for $n \geq 1$.

Let us defer the proof of ($*$) momentarily, and finish the proof of Theorem 11. Pick a distinguished $d \in W(A^\flat)$ such that $W(A^\flat)/d \xrightarrow{\sim} A$. We proved last time that d is a nonzero divisor (since $W(A^\flat)$ is perfect and p-complete). The image of d in $W(B^\flat)$ and $W(C^\flat)$ is also a distinguished element, so modding out by d, we see that $W(R)/d \cong B \otimes^L_A C$. But this means that $B \otimes^L_A C$ is concentrated in degree zero, and there it is the perfectoid ring $W(R)/d$.

Let us now prove ($*$). We will prove a (much) stronger claim (see Bhatt-Scholze's paper on the Witt vector affine Grassmannian): any simplicial commutative $\mathbf{F}_p$-algebra R which is *perfect* is concentrated in degree zero. The universal simplicial commutative $\mathbf{F}_p$-algebra generated by a class in degree $n \geq 0$ is $\mathrm{Sym}(\Sigma^n \mathbf{F}_p)$. It suffices to show that the Frobenius is zero on the homotopy groups of $\mathrm{Sym}(\Sigma^n \mathbf{F}_p)$ for $n > 0$; we will do this by induction on n. First, suppose the claim is true for $\mathrm{Sym}(\Sigma^n \mathbf{F}_p)$. Since $\mathrm{Sym}(\Sigma^{n+1}\mathbf{F}_p) = \mathbf{F}_p \otimes^L_{\mathrm{Sym}(\Sigma^n \mathbf{F}_p)} \mathbf{F}_p$, we see that $\pi_{n+1}\mathrm{Sym}(\Sigma^{n+1}\mathbf{F}_p)$ is Frobenius-equivariantly isomorphic to $\pi_n \mathrm{Sym}(\Sigma^n \mathbf{F}_p)$. This proves the induction step, which reduces us to the case $n = 1$. Then, $\mathrm{Sym}(\Sigma \mathbf{F}_p) = \mathbf{F}_p \otimes^L_{\mathbf{F}_p[t]} \mathbf{F}_p$ for $|t| = 0$. But $\pi_1 \mathrm{Sym}(\Sigma \mathbf{F}_p)$ is the primitives in $\mathbf{F}_p[t]$, i.e., $(t)/(t^2)$. This is killed by Frobenius, as desired. □

Let us now prove Theorem 12. Before doing so, we need a lemma.

Lemma 14. *Let R be a perfectoid ring. Then $\sqrt{pR}$ is flat, and $\sqrt{pR} = \cup_n (\pi^{1/p^n}) = (\pi^{1/p^\infty})$, where $\pi \in R$ is such that $\pi^p = pu$. Moreover, $R[p] = R[\sqrt{pR}]$.*

Proof. We need the following useful observation (which Bhatt calls the "torsion exchange lemma"): if $x, y \in A$ are nonzero divisors, then $A/x[y] = A/y[x]$, because both are π_1 of the Koszul complex associated to $(x, y) \subseteq A$.

Let us now show that $\sqrt{pR} = \cup_n (\pi^{1/p^n})$. First, observe that $\cup_n (\pi^{1/p^n}) \subseteq \sqrt{pR}$, since $(\pi^p) = pR$. We need to see that $R/\cup_n (\pi^{1/p^n})$ is reduced, but this quotient is $R/\pi^{1/p^\infty}$. This is perfect, hence reduced.

Now we show that $\sqrt{pR}$ is flat. Let $M \in \mathcal{D}^{\geq 0}(R)$ be a bounded complex of R-modules. By the long exact sequence in cohomology associated to the distinguished triangle obtained by (derived) tensoring M with $\sqrt{pR} \to R \to R/\sqrt{pR}$, it suffices to show that $M \otimes^L_R R/\sqrt{pR}$ is concentrated in cohomological degrees ≥ -1. But $(M \otimes^L_R R/\sqrt{pR})[1/p] = 0$, so since $\mathbf{Q}_p/\mathbf{Z}_p[-1] \to \mathbf{Z} \to \mathbf{Z}[1/p]$ is a distinguished triangle, we may assume that the cohomology groups of M are p^∞-torsion. In fact, since M is a filtered union of submodules whose cohomology is bounded p^∞-torsion, we may assume M is p^n-torsion. Further reducing to $n = 1$, we can assume M is an R/p-module. Since $W(R^\flat)/d = R$ and d is a nonzero divisor, we see that $W(R/\sqrt{pR}) \otimes^L_{W(R^\flat)} R \cong W(R/\sqrt{pR})/d = R/\sqrt{pR}$. But then $M \otimes^L_R R/\sqrt{pR} = M \otimes^L_{W(R^\flat)} W(R/\sqrt{pR})$. However, since M is killed by p and p is a nonzero divisor in $W(R^\flat)$ and $W(R/\sqrt{pR})$, we see that this is further isomorphic to $M \otimes^L_{R^\flat} R/\sqrt{pR}$.

However, we have already seen that $R/\sqrt{pR} \cong R^\flat/(\pi^{1/p^\infty})$, so it suffices to prove the following more general claim: if B is a perfect $\mathbf{F}_p$-algebra and $x \in B$, then (x^{1/p^∞}) is flat. To see this, it suffices to see that there is an isomorphism

$$\varinjlim (B \xrightarrow{x^{1-\frac{1}{p}}} B \xrightarrow{x^{\frac{1}{p}-\frac{1}{p^2}}} B \to \cdots) \xrightarrow{\cong} (x^{1/p^\infty}),$$

which sends $b \mapsto x^{1/p^n} b$ in the nth spot. (This is sufficient because filtered colimits of flat modules are flat.) It is clear that this map is surjective, so we need to show injectivity. Say $b \in B$ is killed by x^{1/p^n}. Since B is perfect, $x^{1/p^{n+1}} b^{1/p} = 0$, i.e., $x^{1/p^{n+1}} b = 0$. So the transition map kills b, i.e., b vanishes in the direct limit.

Finally, we show that $R[p] = R[\sqrt{pR}]$. Let $d = [a_0] - pu$ with u a unit in $W(R^\flat)$, so that $\sqrt{pR}$ is generated by $[a_0^{1/p^n}]$. Clearly $R[\sqrt{pR}] \subseteq R[p]$, so we need to show that $R[p]$ is killed by $[a_0^{1/p^n}]$ for all n. But $R = W(R^\flat)/d$, so by the torsion exchange lemma,

$$R[p] = (W(R^\flat)/d)[p] = (W(R^\flat)/p)[d] = R^\flat[d].$$

But the image of d in $R^\flat$ is a_0, so we need to show that a_0^{1/p^n} kills $R^\flat[a_0]$. This is a consequence of Lemma 8. □

Proof of Theorem 12. The square under consideration is

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R/\sqrt{pR} \arrow[d] \\ S := R/R[\sqrt{pR}] \arrow[r] & S/\sqrt{pS}. \end{tikzcd} \tag{1}$$

Suppose this square is Cartesian. Then the kernel of the top and bottom horizontal maps agree; but the kernel of the top horizontal map is $\sqrt{pR}$, while the the kernel of the bottom horizontal map is $\sqrt{pS}$. This proves (b). Next, the kernel of the left and right vertical maps agree; but the kernel of the left vertical map is $R[\sqrt{pR}]$, which proves (c). Both (b) and (c) imply that the square is a pushout, so we are reduced to showing that the square is a pullback and that S is p-torsionfree.

To see that S is p-torsionfree, we just need to see that Lemma 14 implies $R[p^\infty] = R[\sqrt{pR}]$. We now show that the square is a pullback. To do this, we will identify each of the perfectoid rings in the four corners as quotients of certain perfect prisms, and show that the square is obtained from a pullback square of perfect prisms. More precisely, let $A = \mathbf{A}_{\mathrm{inf}}(R)$, and let $d \in A$ be a distinguished element such that $R = A/d$. Write $d = [a_0] - pu$, and let $I, J \subseteq R^\flat$ be the ideals defined by $I = (a_0^{1/p^\infty})$ and $J = R^\flat[I]$. Then, we claim that the following square is homotopy Cartesian, and that it agrees with (1):

$$\begin{tikzcd} R = W(R^\flat)/d \arrow[r] \arrow[d] & W(R^\flat/I)/d \arrow[d] \\ W(R^\flat/J)/d \arrow[r] & W(R^\flat/(I+J))/d. \end{tikzcd} \tag{2}$$

Let us first show that this square is homotopy Cartesian. Since modding out by d is the same as derived base-change along $W(R^\flat) \to R$ and d is a nonzero divisor, we are reduced to showing that the following square is homotopy Cartesian:

$$\begin{tikzcd} W(R^\flat) \arrow[r] \arrow[d] & W(R^\flat/I) \arrow[d] \\ W(R^\flat/J) \arrow[r] & W(R^\flat/(I+J)). \end{tikzcd}$$

To check this, it suffices to show the square is homotopy Cartesian mod p^n for each n, and in turn reduce to the case $n = 1$. Mod p, the above square is

$$\begin{tikzcd} R^\flat \arrow[r] \arrow[d] & R^\flat/I \arrow[d] \\ R^\flat/J \arrow[r] & R^\flat/(I+J). \end{tikzcd}$$

To see that this is homotopy Cartesian, first note that the vertical maps in the above diagram are surjective, which means that being homotopy Cartesian is equivalent to being a pullback square. We know that the pullback $R^\flat/J \times_{R^\flat/(I+J)} R^\flat/I$ is $R^\flat/(I \cap J)$, so we need to see that $I \cap J = 0$. So let $x \in I \cap J$; then, $xI = 0$ (because $x \in J$). Since $x \in I$, we see that $x^2 = 0$, i.e., $x^p = 0$. This means $x = 0$, since R is perfect.

Finally, we identify (1) with (2).

- There is an isomorphism $W(R^\flat/I)/d \cong R/\sqrt{pR}$. Indeed, since $a_0 \in I$, we know that $(d) = (p) \in W(R^\flat/I)$. Therefore, $W(R^\flat/I)/d \cong R^\flat/I = R^\flat/(a_0^{1/p^\infty})$. This is isomorphic to $R/\sqrt{pR}$ by Lemma 14, as desired.
- There is an isomorphism $W(R^\flat/J)/d \cong S$. Let us write $S' = W(R^\flat/J)/d$; then, we claim that S' is p-torsionfree (so that the map $R \to S'$ factors through a map $R/R[p^\infty] = S \to S'$). We see from the torsion exchange lemma in the proof of Lemma 14 that
$$S'[p] = (W(R^\flat/J)/d)[p] \cong (W(R^\flat/J)/p)[d] = (R^\flat/J)[d].$$
We therefore wish to show that $(R^\flat/J)[d] = 0$. The image of d in $R^\flat/J$ is a_0, so we need to see that a_0 is a nonzero divisor in $R^\flat/J$. Suppose $b \in R^\flat/J$ is killed by a_0. Choose a lift $\widetilde{b} \in R^\flat$ which lifts b; then, $a_0\widetilde{b} = c$ for some $c \in R^\flat[a_0^{1/p^\infty}]$. But then $0 = a_0 c = a_0^2\widetilde{b}$, so $\widetilde{b} \in R^\flat[a_0^2]$. We have already seen in Lemma 8 that $R^\flat[a_0^2] = R^\flat[a_0^{1/p^\infty}]$, so $\widetilde{b} \in R^\flat[a_0^{1/p^\infty}]$, i.e., $b = 0 \in R^\flat/J$.

  Since S' is p-torsionfree, we get a map $S \to S'$. We claim that this map is an isomorphism. For this, we need to see that the kernel of $R \to S'$ is in $R[p^\infty]$. However, (2) is a pullback, so the kernel of $R \to S'$ is contained in $\overline{R} = W(R^\flat/I)$. But we've already seen that $\overline{R}$ is of characteristic p, proving the desired claim.
- There is an isomorphism $W(R^\flat/(I+J))/d \cong S/\sqrt{pS}$. For this, one argues in the same way as in the second bullet.

□

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