ÉTALE FUNDAMENTAL GROUPS

The goal of this talk is to give an introduction to the étale fundamental group. We begin with some motivation from topology, and then proceed to study the appropriate algebraic analogue of the fundamental group.

1. The classical fundamental group

Let X be a topological space which is path-connected, locally path-connected, and locally simply-connected, and fix a basepoint $x \in X$. The datum of the basepoint will be crucial.

Definition 1.1. The universal cover $\widetilde{X}$ of X is the initial path-connected and simply-connected pointed space (Y, y) equipped with a covering map $\pi : Y \to X$ which is pointed (in the sense that $\pi(y) = x$).

Example 1.2. Let $X = \mathbf{C}^\times$ be the punctured plane (and pick any point to be the basepoint). Then the universal cover $\widetilde{X}$ can be chosen to be $\mathbf{C}$, and the covering $\pi : \widetilde{X} \to X$ is given by the exponential. Note that the map π is *not* finite: the fiber over any point $z \in \mathbf{C}^\times$ is $\mathbf{Z}$.

Remark 1.3. Under the above assumptions on X, the universal cover $\widetilde{X}$ exists: one explicit model is as the space of homotopy classes of paths $\gamma : [0,1] \to X$ such that $\gamma(0) = x$. The covering map $\pi : \widetilde{X} \to X$ sends γ to $\gamma(1)$.

The universal cover, being defined as the initial object in a certain category, admits nice categorical properties (even though the topological space might be quite large).

Definition 1.4. Let $\mathcal{C}_X$ denote the category of covering spaces $\pi : Y \to X$ with finitely many connected components, and let $\mathcal{C}_X^{\mathrm{fin}}$ denote the category of *finite* covering spaces $\pi : Y \to X$ (i.e., the fibers are finite). Note that the universal cover is generally *not* in $\mathcal{C}_X^{\mathrm{fin}}$, as illustrated by Example 1.2.

Construction 1.5. There is a functor $F : \mathcal{C}_X \to \mathrm{Set}$, sending a covering space $\pi : Y \to X$ to $\pi^{-1}(x)$. Note that $\mathcal{C}_X^{\mathrm{fin}} \simeq \mathcal{C}_X \times_{\mathrm{Set}} \mathrm{Fin}$, where $\mathrm{Fin} \subseteq \mathrm{Set}$ is the full subcategory spanned by the finite sets.

We leave the following proposition as an exercise to the reader; the key point is to observe that it is equivalent to the universal property of the universal cover.

Proposition 1.6. *The functor $F : \mathcal{C}_X \to \mathrm{Set}$ is representable by $\widetilde{X}$.*

Corollary 1.7. *Let $\pi : Y \to X$ be an object of $\mathcal{C}_X$. Then there is an action of $\mathrm{Aut}_X(\widetilde{X})$ on $\pi^{-1}(x) \subseteq Y$.*

Exercise 1.8. There is an isomorphism $\mathrm{Aut}_X(\widetilde{X}) \cong \mathrm{Aut}(F)$ (where the latter group consists of invertible natural transformations from F to itself).

The following is sometimes regarded as a theorem in point-set topology.

Definition 1.9. The fundamental group $\pi_1(X, x)$ is the automorphism group $\mathrm{Aut}_X(\widetilde{X})$ in $\mathcal{C}_X$. By Remark 1.3, the fundamental group $\pi_1(X, x)$ may also be regarded as the group of homotopy classes of loops based at x.

Exercise 1.10. It is perhaps enlightening to understand how the above definition of the fundamental group relates to the classical result (called the Galois correspondence) that there is an order-reversing equivalence (i.e., a contravariant equivalence of posets) between conjugacy classes of subgroups of $\pi_1(X,x)$ and path-connected (pointed) covering spaces of X. More precisely:

Theorem 1.11. *The functor $F : \mathcal{C}_X \to \mathrm{Set}$ refines to a functor $F' : \mathcal{C}_X \to \mathrm{Rep}_{\mathrm{Set}}(\pi_1(X,x))$ (which means the category of $\pi_1(X,x)$-sets). The Galois correspondence states that the functor F' is an equivalence.*

We can also consider a variation of the fundamental group where we only consider *finite* covering spaces of X (i.e., where we work with $\mathcal{C}_X^{\mathrm{fin}}$ instead of $\mathcal{C}_X$).

Definition 1.12. Let $F^{\mathrm{fin}} : \mathcal{C}_X^{\mathrm{fin}} \to \mathrm{Set}$ denote the restriction of F to the inclusion $\mathcal{C}_X^{\mathrm{fin}} \subseteq \mathcal{C}_X$. The profinite fundamental group $\hat{\pi}_1(X,x)$ of the pointed space X is the automorphism group $\mathrm{Aut}(F^{\mathrm{fin}})$.

Remark 1.13. The profinite fundamental group acquires its name for a good reason: if $G \le \pi_1(X,x)$ is a subgroup of finite index, then the Galois correspondence associates to G a finite pointed covering space $Y \to X$. If G is normal, then the group of pointed automorphisms of Y over X is $\pi_1(X,x)/G$, and there are canonical maps $\pi_1(X,x) \to \pi_1(X,x)/G$. Taking the inverse limit over the poset of all normal subgroups G of finite index defines the profinite completion of $\pi_1(X,x)$. The resulting group is isomorphic to the profinite fundamental group.

Notation 1.14. Let G be a *profinite* group. We will denote by $\mathrm{Rep}_{\mathrm{FinSet}}(G)$ the category of *finite* G-sets, i.e., finite sets equipped with a continuous action of G. This condition translates to asking that the action of G on the open set factors through the quotient by some open subgroup of finite index.

Theorem 1.15. *The functor $F : \mathcal{C}_X^{\mathrm{fin}} \to \mathrm{Fin}$ refines to a functor $F' : \mathcal{C}_X^{\mathrm{fin}} \to \mathrm{Rep}_{\mathrm{FinSet}}(\pi_1(X,x)) \simeq \mathrm{Rep}_{\mathrm{FinSet}}(\widehat{\pi}_1(X,x))$. The Galois correspondence states that the functor F' is an equivalence.*

2. Abstract Galois theory

Motivated by the examples of §1, we now embark on a quest to answer the following question.

Question 2.1. Let $\mathcal{C}$ be a category. Under what conditions is $\mathcal{C}$ equivalent to the category $\mathrm{Rep}_{\mathrm{FinSet}}(G)$ of finite G-sets for some profinite group G?

Question 2.1 is not very good as stated: the category $\mathrm{Rep}_{\mathrm{FinSet}}(G)$, being defined by the relationship between G and sets, is canonically equipped with a forgetful functor $\mathrm{Rep}_{\mathrm{FinSet}}(G) \to \mathrm{Fin}$. We can therefore refine Question 2.1 to the following:

Question 2.2. Let $\mathcal{C}$ be a category equipped with a functor $F : \mathcal{C} \to \mathrm{Fin}$. Under what conditions on $\mathcal{C}$ and F is there an equivalence $\mathcal{C} \simeq \mathrm{Rep}_{\mathrm{FinSet}}(G)$ of categories over Fin?

To answer Question 2.2, it is useful to understand properties of the category $\mathrm{Rep}_{\mathrm{FinSet}}(G)$.

Example 2.3. Every finite G-set is isomorphic to a finite coproduct $\coprod_{i=1}^{n} G/H_i$ where $H_i \le G$ is a subgroup of finite index. The action of G on each G/H_i is transitive. The functor $F : \mathrm{Rep}_{\mathrm{FinSet}}(G) \to \mathrm{Set}$ is also conservative, meaning that a map $x \to y$ in $\mathrm{Rep}_{\mathrm{FinSet}}(G)$ is an equivalence of G-sets if and only if $F(x) \to F(y)$ is an equivalence. The category $\mathrm{Rep}_{\mathrm{FinSet}}(G)$ also has limits and colimits, and the forgetful functor preserves these finite limits and colimits.

Definition 2.4. A category $\mathcal{C}$ equipped with a functor $F : \mathcal{C} \to \mathrm{Fin}$ is said to be *Galois* if it satisfies the following conditions.

(a) $\mathcal{C}$ has finite limits and finite colimits.

(b) The functor F preserves finite limits and finite colimits, and is conservative (i.e., detects equivalences).
(c) Say that an object $x \in \mathcal{C}$ is connected if x is not initial, and x is irreducible (meaning that if $x = y \sqcup z$, then one of the maps $y \to x$ or $z \to x$ is an equivalence). Then every object $c \in \mathcal{C}$ can be written as a finite coproduct of connected objects of $\mathcal{C}$.

The main result is the following.

Theorem 2.5. *A category $\mathcal{C}$ equipped with a functor $F : \mathcal{C} \to \mathrm{Fin}$ is Galois if and only if there is a profinite group G and an equivalence $\mathcal{C} \simeq \mathrm{Rep}_{\mathrm{FinSet}}(G)$ of categories over* Fin*. Moreover, $G \cong \mathrm{Aut}(F)$.*

We will defer a sketch of a proof and further discussion of Theorem 2.5 to the end, and discuss the example that we will be interested in.

Definition 2.6. Let X be a connected scheme over a field k. Fix a geometric point $x : \mathrm{Spec}(\overline{k}) \to X$. Let FEt_X denote the category of finite étale covers $\pi : Y \to X$, and let $F : \mathrm{FEt}_X \to \mathrm{Set}$ denote the functor sending $\pi : Y \to X$ to the set $Y_x := \mathrm{Map}_X(x, Y)$ (i.e., the set of geometric points of Y over x with residue field $k(x)$). In other words, consider the fiber product

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] & Y \arrow[d, "\pi"] \\ \mathrm{Spec}(\overline{k}) \arrow[r, "x"'] & X; \end{tikzcd}$$

the scheme Y_x is finite étale over $\mathrm{Spec}(\overline{k})$, and hence is a disjoint union of finitely many copies of $\mathrm{Spec}(\overline{k})$.

Exercise 2.7. The notion of a finite étale cover is a good algebraic generalization of the notion of a finite covering: the geometric fiber over each geometric point of X has the same number of points. Moreover, for each point $x \in X$, there is an étale neighborhood over which π is trivial (i.e., a disjoint union of varieties isomorphic to the neighborhood).

Proposition 2.8. *The category FEt_X and the functor $F : \mathrm{FEt}_X \to \mathrm{Set}$ is Galois.*

Proof sketch. We will check that each of the conditions for being Galois is satisfied.

(a) The category FEt_X has finite limits if it has products and equalizers, or, equivalently, pullbacks and a terminal object. The terminal object is the identity $X \to X$, and pullbacks are given by fiber products (note that fiber products of finite étale morphisms are finite étale).
Next, FEt_X has finite colimits if it has coproducts and coequalizers. Coproducts certainly exist: they are given by the disjoint union (which remain finite étale over X). For coequalizers, we give the following sketch: if $\pi : Y \to X$ and $\pi' : Z \to X$ are finite étale covers, and $f, g : Y \to Z$ are maps over X, let $\mathcal{F}$ denote the equalizer of $f, g : \pi'_*\mathcal{O}_Z \to \pi_*\mathcal{O}_Y$. By passing to a finite étale cover of X, we may assume that $\pi'_*\mathcal{O}_Z$ and $\pi_*\mathcal{O}_Y$ are finite products of $\mathcal{O}_X$, so $\pi'_*\mathcal{O}_Z = \prod_J \mathcal{O}_X$ and $\pi_*\mathcal{O}_Y = \prod_I \mathcal{O}_X$. The morphisms f and g are determined by maps $f, g : J \to I$. The equalizer of f, g defines a finite set I', and $\mathcal{F} \cong \prod_{I'} \mathcal{O}_X$.
(b) The functor $F : \mathrm{FEt}_X \to \mathrm{Fin}$ preserves finite limits and colimits, and is conservative. We first argue that F is conservative. Let $Y \to Z$ be a map in FEt_X which induces a bijection $F(Y) \xrightarrow{\sim} F(Z)$; we need to show that $Y \xrightarrow{\sim} Z$. Using (c) below, we can assume that Z is connected. The map $Y \to Z$ is finite étale and therefore finite locally free. It therefore suffices to show that the map $Y \to Z$ is of degree 1, i.e., the preimage of any

geometric point of Z is a singleton. Since the map $F(Y) \xrightarrow{\sim} F(Z)$ is a bijection, this is true in a neighborhood of any point of Z over $x \in X$. The degree is locally constant, so by connectedness of Z we see that the fiber of $Y \to Z$ over any geometric point of Z must be a singleton, as desired.

We now sketch an argument that F preserves finite limits and colimits, and leave the details to the reader. First, observe that there is an equivalence $\mathrm{FEt}_{\mathrm{Spec}(\overline{k})} \simeq \mathrm{Fin}$. Next, show that if $X_2 \to X_1$ is a morphism, then the base-change functor $\mathrm{FEt}_{X_1} \to \mathrm{FEt}_{X_2}$ preserves both finite limits and finite colimits. Finally, show that the functor $F : \mathrm{FEt}_X \to \mathrm{Fin}$ is given by base-change along $x : \mathrm{Spec}(\overline{k}) \to X$ along with the identification $\mathrm{FEt}_{\mathrm{Spec}(\overline{k})} \simeq \mathrm{Fin}$.

(c) The connected objects of FEt_X are the connected schemes, and so decomposing a finite étale cover of X into its connected components yields the desired condition.

□

Theorem 2.5 and Proposition 2.8 imply:

Corollary 2.9. *Let* $\widehat{\pi}_1^{\mathrm{et}}(X,x)$ *denote the automorphism group* $\mathrm{Aut}(F : \mathrm{FEt}_X \to \mathrm{Fin})$. *Then there is an equivalence* $\mathrm{FEt}_X \simeq \mathrm{Rep}_{\mathrm{FinSet}}(\widehat{\pi}_1^{\mathrm{et}}(X,x))$.

3. Examples of the étale fundamental group

Definition 3.1. The profinite group $\widehat{\pi}_1^{\mathrm{et}}(X,x)$ is called the étale fundamental group of X.

Example 3.2. Let $X = \mathrm{Spec}(k)$, and let x be a geometric point of X representing a fixed algebraic closure $\overline{k}$ of k. Every finite étale cover X is a disjoint union of the spectra of finite separable extensions of k, and so (since separable Galois extensions can be identified with normal subgroups of the Galois group of a separable closure k^{sep} of k) we find that $\widehat{\pi}_1^{\mathrm{et}}(\mathrm{Spec}(k),\overline{k}) \cong \mathrm{Gal}(k^{\mathrm{sep}}/k)$. For instance, the étale fundamental group of $\mathbf{Q}$ is the absolute Galois group of $\mathbf{Q}$, while the étale fundamental group of $\mathbf{F}_p$ is $\widehat{\mathbf{Z}}$, generated by the Frobenius.

We would like to compare the étale fundamental group to the topological fundamental group. A comparison is given by the following.

Theorem 3.3 (Grothendieck-Riemann existence theorem)**.** *Let X be a $\mathbf{C}$-scheme which is locally of finite type. Then the $\mathbf{C}$-points functor* $\mathrm{FEt}_X \to \mathcal{C}^{\mathrm{fin}}_{X(\mathbf{C})}$ *is an equivalence of categories.*

Exercise 3.4. Is it true that the functor from the category of *all* étale covers of X to covering spaces of $X(\mathbf{C})$ is an equivalence?

Corollary 3.5. *There is an isomorphism* $\widehat{\pi}_1^{\mathrm{et}}(X,x) \cong \widehat{\pi}_1(X(\mathbf{C}),x)$ *of profinite groups.*

The goal for the remainder of this section is to study some examples. Corollary 3.5 already presents us with a lot of computable examples.

Example 3.6. We will work over $\mathbf{C}$.

(a) Let $X = \mathbf{A}^1$, and let $x \in X$ be any geometric point. Then $X(\mathbf{C}) \cong \mathbf{C}$, and so $\widehat{\pi}_1^{\mathrm{et}}(\mathbf{A}^1,x) = \widehat{\pi}_1(\mathbf{C},x) = 0$.

(b) Similarly, let $X = \mathbf{P}^1$, and let $x \in X$ be any geometric point. Then $\mathbf{P}^1(\mathbf{C}) \cong S^2$, and so $\widehat{\pi}_1^{\mathrm{et}}(\mathbf{P}^1,x) = \widehat{\pi}_1(S^2,x) = 0$.

(c) More generally, if $X = \mathbf{P}^1 - [n]$, where $[n] \subseteq \mathbf{P}^1$ is a collection of $n \geq 1$ points, then $X(\mathbf{C}) \cong \mathbf{C} - [n-1]$. It follows that $\widehat{\pi}_1^{\mathrm{et}}(X,x)$ is isomorphic to a free profinite group on $n-1$ generators.

The following is useful.

Proposition 3.7. *Let X be a connected and geometrically connected algebraic variety over a field k. Then there is an exact sequence*

$$0 \to \widehat{\pi}_1^{\mathrm{et}}(X_{k^{\mathrm{sep}}}, x) \to \widehat{\pi}_1^{\mathrm{et}}(X, x) \to \mathrm{Gal}(k^{\mathrm{sep}}/k) \to 0.$$

The maps are induced by pulling back étale covers along the canonical maps $X_{k^{\mathrm{sep}}} \to X \to \mathrm{Spec}(k)$.

Remark 3.8. Rather than explaining the proof, let us give an intuitive explanation for this result in the case when the exact sequence splits (which occurs, for instance, if X has a k-rational point). In this case, the proposition says that a finite $\widehat{\pi}_1^{\mathrm{et}}(X, x)$-set can be thought of a finite $\widehat{\pi}_1^{\mathrm{et}}(X_{k^{\mathrm{sep}}}, x)$-set along with a compatible $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-action. Translating along the equivalence $\mathrm{FEt}_X \simeq \mathrm{Rep}_{\mathrm{FinSet}}(\widehat{\pi}_1^{\mathrm{et}}(X, x))$, this says that finite étale covers of X are the same as finite étale covers of $X_{k^{\mathrm{sep}}}$ equipped with a compatible $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-action. In this case, the proposition may therefore be regarded as a version of Galois descent.

Example 3.9. Let $k = \mathbf{R}$, and let $k^{\mathrm{sep}} = \mathbf{C}$. Then Example 3.6 implies:

$$\widehat{\pi}_1^{\mathrm{et}}(\mathbf{A}^1_{\mathbf{R}}, x) \cong \mathrm{Gal}(\mathbf{C}/\mathbf{R}) = \mathbf{Z}/2, \ \widehat{\pi}_1^{\mathrm{et}}(\mathbf{P}^1_{\mathbf{R}}, x) \cong \mathrm{Gal}(\mathbf{C}/\mathbf{R}) = \mathbf{Z}/2.$$

Example 3.10. The étale fundamental group of the affine line is *not* zero over a base field of nonzero characteristic. Indeed, let k be an algebraically closed field of characteristic $p > 0$, and consider the Artin-Schreier map $\alpha : \mathbf{A}^1 \to \mathbf{A}^1$ given by $t \mapsto t^p - t$. Since

$$\frac{d}{dt}\alpha(t) = pt^{p-1} - 1 = -1,$$

we see that α is indeed étale. Since k is algebraically closed, α is surjective, too. Observe that α is a group homomorphism, and so the Galois group of this covering space is given by the kernel of α. It is easy to see that $\ker(\alpha) \cong \mathbf{Z}/p$, and so $\widehat{\pi}_1^{\mathrm{et}}(\mathbf{A}^1_k, x)$ is nontrivial (it has a quotient $\mathbf{Z}/p$).

Example 3.11. The étale fundamental group of $\mathbf{P}^1$ over any algebraically closed base field is still zero. Indeed, the Riemann-Hurwitz theorem holds for finite separable morphisms of curves over any such field (see §IV.2.4 of Hartshorne), so we can argue as follows. Let $f : C \to \mathbf{P}^1$ be a connected finite étale cover of $\mathbf{P}^1$ of degree n, where C is a curve of genus g. Since f is unramified, the ramification divisor is zero. Therefore, by Riemann-Hurwitz, we see:

$$2g - 2 = n(2g(\mathbf{P}^1) - 2) = -2n.$$

This occurs if and only if $g = 0$ and $n = 1$, i.e., $C = \mathbf{P}^1$.

Example 3.12. The étale fundamental group of $\mathbf{P}^n$ over any algebraically closed base field is still zero. Indeed, we can argue by induction, with the base case given by Example 3.11. Consider a connected finite étale cover $f : X \to \mathbf{P}^n$, and let $H \subseteq \mathbf{P}^n$ be a hyperplane. The pullback of H along f is a divisor on X, which is ample because f is finite. Since f^*H is ample, it is connected, and so f defines a connected finite étale cover $f' : f^*H \to H$ of H. The inductive hypothesis implies that f' is an isomorphism, so $\deg(f) = 1$, i.e., f is an isomorphism.

Example 3.13. Let X be a normal Noetherian scheme, and let K denote its function field. Let Ω denote a separably closed field containing K, and let $x : \mathrm{Spec}(\Omega) \to X$ denote the resulting geometric point of X. Denote by L the union of all finite separable extensions K' of K in Ω which are unramified over X (i.e., such that the normalization of X in K' is étale over X).

Proposition 3.14. *In the above setup, there is an isomorphism $\widehat{\pi}_1^{\mathrm{et}}(X, x) \cong \mathrm{Gal}(L/K)$.*

Proof sketch. We follow Tag 0BQM. It suffices to show that every connected finite étale cover $f : Y \to X$ arises as the normalization of X in the finite separable extension K'/K, where K' is the function field of Y. Namely, we need to show that Y is the normalization of X in the function field of Y. First, recall that normality is local in the étale topology, and so Y is itself normal. Moreover, Y is integral (because it has a finite number of irreducible components). Let Y' denote the normalization of X in the function field $k(Y)$ of Y, so there is a map $Y' \to X$ which factors as $Y \to Y' \to X$ which exhibits Y' as the normalization of Y in $k(Y)$[1]. We now appeal to Tag 0AB1, which says that any finite birational morphism $X_1 \to X_2$ of integral schemes with X_2 normal is an isomorphism. Applying this to the morphism $Y \to Y'$ shows that $Y \to Y'$ is an isomorphism. □

Example 3.15. Let K be a global field, and S a finite set of places. Let $\mathcal{O}_{K,S}$ denote the ring of S-integers in K. The above proposition implies that $\widehat{\pi}_1^{\mathrm{et}}(\mathrm{Spec}(\mathcal{O}_{K,S}), x)$ is isomorphic to $\mathrm{Gal}(K_S/K)$, where K_S is the maximal Galois extension of K which is unramified over S.

4. A proof sketch for Theorem 2.5

The goal of this section is to sketch a proof of Theorem 2.5. We first need to define the profinite structure on $\mathrm{Aut}(F)$.

Construction 4.1. Any automorphism of the functor $F : \mathcal{C} \to \mathrm{Fin}$ defines an automorphism of $F(x)$ for $x \in \mathcal{C}$. There is therefore a homomorphism

$$\mathrm{Aut}(F) \to \prod_{x \in \mathcal{C}} \mathrm{Aut}(F(x)).$$

It is easy to see that this homomorphism is injective, and so $\mathrm{Aut}(F)$ can be regarded as a subgroup of the product group. Since each $F(x)$ is finite, the group $\mathrm{Aut}(F(x))$ is also finite. The product group is generally not finite, but it acquires a topology if one equips each $\mathrm{Aut}(F(x))$ with the discrete topology. One can show that $\mathrm{Aut}(F)$ is then a *closed* subgroup of the product group, and so is a profinite group.

Proof sketch of Theorem 2.5. For details, the reader should consult Tag 0BMQ. The assumption that F is conservative implies that it is faithful. Indeed, suppose $f, g : x \to y$ are two morphisms in $\mathcal{C}$ such that $F(f) = F(g)$. If z is the equalizer of f and g, this implies that $F(z) = F(x)$. Because F is conservative, we see that $z = x$, and so $f = g$.

It remains to prove that F is essential surjective and full. Before doing so, we need an important result, for which we refer the reader to Tags 0BN2 and 0BN3. Say that an object $x \in \mathcal{C}$ is Galois if it is connected, and $\mathrm{Aut}(x)$ acts transitively on x. Then:

- (∗) If $x \in \mathcal{C}$ is a connected object, then there is a Galois object $y \in \mathcal{C}$ with a morphism $y \to x$. This can be shown to imply that $\mathrm{Aut}(F)$ acts transitively on $F(x)$.

We may now prove that F is essential surjective and full.

- (a) For essential surjectivity, recall that if G is a profinite group, then all finite G-sets are disjoint unions of orbits G/H with H an open subgroup of G of finite index. Since F preserves finite colimits, it therefore suffices to lift each G/H to $\mathcal{C}$, where $G = \mathrm{Aut}(F)$. By Construction 4.1, there is a finite set $\{x_i\}$ of (connected) objects of $\mathcal{C}$ such that H contains the kernel K of the map $G \to \prod_i \mathrm{Aut}(F(x_i))$.

  By (∗), there is a Galois object $y \in \mathcal{C}$ along with a map from y to a connected component of $\prod_i x_i$. Since y is connected, $\mathrm{Aut}(F)$ acts transitively on $F(y)$, there is an open subgroup $U \subseteq G$ such that $F(y) = G/U$ in $\mathrm{Rep}_{\mathrm{FinSet}}(G)$. In order to show that G/H is in the image of F, it therefore suffices to show two things:
  - The group U is contained in H, and is a normal subgroup.
  - The (opposite of the) quotient group H/U acts on y.

[1] Have I got the direction of the map wrong? According to https://mathoverflow.net/a/46401, the map should go the other way: the normalization of X is the initial object in the category of finite birational maps to X. In particular, there should be a map $Y' \to Y$.

Indeed, if we now let x denote the quotient of y by H/U (i.e., the coequalizer of all arrows $h : y \to y$ for $h \in H/U$), then the fact that F preserves finite colimits implies that

$$F(x) = F(y)/(H/U) \cong G/H,$$

as desired.

We now prove the two bullets. For the first bullet, note that because each x_i is connected, we see that $F(y)$ surjects onto $F(x_i)$, which implies that $U \subseteq K$, i.e., $U \subseteq H$. It remains to show that U is normal in H. Since y is Galois, we also see that $\mathrm{Aut}(y)$ acts transitively on $F(y) \cong G/U$; because $\mathrm{Aut}(y) \cong \mathrm{Aut}_{\mathrm{Rep}_{\mathrm{FinSet}}(G)}(G/U)$, this implies that U is normal.

We now turn to the second bullet. Since $F(y) = G/U$, and U is normal in G, we see that $\mathrm{Aut}(y)$ is isomorphic to the opposite of the group G/U. Since H/U is a subgroup of G/U, we now see that the opposite of H/U is a subgroup of $\mathrm{Aut}(y)$, i.e., the opposite of H/U acts (faithfully!) on y, as desired.

(b) To prove that F is full, suppose $f : F(x) \to F(y)$ is a map in $\mathrm{Rep}_{\mathrm{FinSet}}(G)$, where $x, y \in \mathcal{C}$. We need to define a map $f_0 : x \to y$ such that $F(f_0) = f$. The map f is determined by its graph $\Gamma \subseteq F(x) \times F(y) \cong F(x \times y)$. The graph is a union of connected components of $F(x) \times F(y)$, which can be lifted along F; in other words, Γ is the image under F of a union z of connected components of $x \times y$. The map $\Gamma \cong F(z) \to F(x)$ is bijective; since F is conservative, the map $z \to x$ must be an equivalence in $\mathcal{C}$. It follows that f is the image under F of the composite $x \cong z \to y$, as desired.

□

Remark 4.2. It turns out that Theorem 2.5 is (almost) a special case of a much more general result, known as the Barr-Beck monadicity theorem. This result states that a functor $F : \mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{C}$ as the category of algebras for a monad on $\mathcal{D}$ if and only if F has a left adjoint, is conservative, coequalizers which split under F do in fact exist in $\mathcal{C}$, and F preserves such coequalizers. Another special case of the Barr-Beck theorem that might be familiar is the Tannakian formalism, which states that any abelian symmetric monoidal category where all objects are dualizable, equipped with an exact and conservative functor to Vect_k (with k a field of characteristic zero) is equivalent to the category of representations of an algebraic group over k.

We will not prove Theorem 2.5 via the Barr-Beck theorem in these notes, but the idea of the argument is as follows: by enlarging $\mathcal{C}$ and Fin to the associated *pro-categories*, one produces a functor $\mathrm{Pro}(F) : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathrm{Fin})$. Since F is conservative and preserves finite limits and finite colimits, one can show that $\mathrm{Pro}(F)$ in fact satisfies the conditions of the Barr-Beck theorem. In other words, there is a monad defined on Pro(Fin) such that $\mathrm{Pro}(\mathcal{C})$ is the category of algebras for this monad. With a little more work, one can identify this monad as the free algebra functor for a group object in Pro(Fin), i.e., for a profinite group.

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