Title: Overview of the Lawrence-Venkatesh proof

1. §1: Mordell Conjecture

Notation 1 (for whole talk)**.**

- $K \subset \mathbb{C}$, a number field.
- C/K, a smooth proper curve of genus g.
- $D \subset C$, a possibly empty divisor
- $U := C - D$
- S, a finite set of places in K which contains all infinite places
- $\mathscr{O} = \mathscr{O}_{K,S}$, the ring of S-integers.
- $\mathcal{U}/\mathscr{O}$, a smooth, proper model of U.

♠♠♠ Niven: [Faltings '83, Siegel '29, Mordell '22]

Theorem 2 (Faltings–Siegel, née Mordell Conjecture)**.** *Let $\chi(U) = 2 - 2g - \deg D$ be the (topological) Euler characteristic of $U(\mathbb{C})$. Then,*

$$\chi(U) < 0 \implies \#\mathcal{U}(\mathscr{O}) < \infty.$$

Remark 3.

- If $D = 0$, then $U = C, \mathcal{U}(\mathscr{O}) = C(K)$, and $\chi(U) < 0 \iff g \geq 2$. This case is the Mordell Conjecture:

$$g \geq 2 \implies \#C(K) < \infty.$$

  {☐ *In fact, it turns out that this case implies the statement above. Any curve with negative Euler characteristic has a f.étale cover of genus ≥ 2* ☐}
- More generally, $\chi(U) < 0 \iff U$ looks like

$$\mathbb{P}^1 - \{\geq 3 \text{ points}\}, \quad \underbrace{E}_{g=1} - \{\geq 1 \text{ point}\}, \quad \text{or} \quad \underbrace{C}_{g\geq 2} - \{\geq 0 \text{ points}\}.$$ ○

Goal (of the seminar). Give Lawrence–Venkatesh's proof of Mordell's conjecture.

{☐ *They actually first handle the case of $\mathbb{P}^1$ minus 3 points as a warmup. So the talks after this one will look at this case before we jump into the details of handle curves of genus ≥ 2.* ☐}

2. §2: Lawrence–Venkatesh Proof

Setup 4. Assume we have a smooth projective family $\pi\colon X \to U$ over K, which further extends to a smooth projective $\pi\colon \mathcal{X} \to \mathcal{U}$ over $\mathscr{O}$. {☐ *This is always possibly after possibly enlarging S* ☐} We also fix a 'suitable' prime p away from S. ♠♠♠ Niven: [e.g. odd and unramified in K]

Remark 5. Given $y \in \mathcal{U}(\mathscr{O})$, we can consider the fiber $X_y/\mathscr{O}$. Its étale cohomology $\mathrm{H}^n_{\text{ét}}(X_{y,\overline{K}}, \mathbb{Q}_p)$ is a representation of $G_K := \mathrm{Gal}(\overline{K}/K)$, so we obtain an assignment

$$\Psi\colon \mathcal{U}(\mathscr{O}) \longrightarrow \{\mathbb{Q}_p\text{-reps of } G_K\}.$$

To prove finiteness of $\mathcal{U}(\mathscr{O})$, it suffices to show this map has finite image and finite fibers. {☐ *The point is that $C(K)$ or $\mathcal{U}(\mathscr{O})$ is a set, so it doesn't have much algebraic structure to exploit. Whereas, Galois reps have lots of additional algebraic structure.* ☐} ○

Example 6. Say $U = C$ is a proper curve. Fix a finite group G. Then, for any $y \in C(K)$, there are only finitely many Galois G-curves $\{X_{y,i} \to C\}_i$ which are each unramified away from p. There exists a family $X \to C$ whose fiber above y is $X_y = \bigsqcup_i X_{y,i}$. In this setting, the map Ψ is

$$\Psi(y) = \bigoplus_i T_p \operatorname{Jac}(X_{y,i}).$$

♠♠♠ Niven: [Really, the summands should be $\left(T_p \operatorname{Jac}(X_{y,i}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\right)^\vee \simeq T_p \operatorname{Jac}(X_{y,i}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(-1)$, but who wants to write all that?] {□ *So you can think of* Ψ *as keeping track of how Tate modules of Jacobians of fibers vary as you move along the base.* □} △

{□ *So we attach a Galois rep to every point of our curve. We want to know that only f.many representations arise in this way and that the representations vary enough so that the fibers of this map are finite.* □}

2.1. **§2.1: Ψ has finite image.** {□ *We'll first explain why the* $\operatorname{im}\Psi$ *has only f.many semisimple representations. This will just leave handling the* y *for which* $\Psi(y)$ *is not semisimple.* □}

Fact (Consequence of Weil conjectures)**.** Let $d := \dim \mathrm{H}^q_{\text{ét}}(X_{y,\overline{K}}, \mathbb{Q}_p)$ (independent of y). Then, for any $y \in \mathcal{U}(\mathscr{O})$, the Galois rep

$$\Psi(y)\colon G_K \longrightarrow \mathrm{GL}_d(\mathbb{Q}_p)$$

is unramified outside S and `pure of weight` n in the sense that every eigenvalue of $\Psi(y)(\mathrm{Frob}_v)$, for any $v \notin S$, is an algebraic integer all of whose conjugates have complex absolute value $q_v^{n/2}$, where $q_v = \#\kappa(v)$.

Theorem 7 (Faltings)**.** *The set of equivalence classes of* semisimple *d-dimensional $\mathbb{Q}_p$-representations of G_K which are unramified outside S and pure of weight n is finite.*

Proof.

- Step 1: for any $N \geq 1$, there exists a finite set T_N of primes of K s.t. $T_N \cap S = \emptyset$ and for any continuous map $\chi\colon G_K \to G$ unramified outside S to a group G of order $\leq N$,
$$\operatorname{im}\chi = \{\chi(\mathrm{Frob}_v)\colon v \in T_N\}.$$
For each χ, let $L_\chi = \overline{K}^{\ker\chi}$. This field is unramifed outside S and of degree $\leq N$, so Hermite's theorem implies that there are only f.many possibilities for L_χ. Let L be the compositum of all such L_χ's, so L/K is finite and each χ factors through $\mathrm{Gal}(L/K)$. The result follows from Chebotarev density.
- Step 2: There exists a finite set T of primes of K s.t. $T \cap S = \emptyset$ and, for any two d-dimensional $\mathbb{Q}_p$-reps ρ, ρ' of G_K which are unramified outside S,
$$\operatorname{Tr}\rho(\mathrm{Frob}_v) = \operatorname{Tr}\rho'(\mathrm{Frob}_v) \text{ for all } v \in T \implies \operatorname{Tr}\rho(g) = \operatorname{Tr}\rho'(g) \text{ for all } g \in G_K.$$
Set $T = T_{p^{2d^2}}$. Wlog ρ, ρ' take values in $\mathrm{GL}_d(\mathbb{Z}_p)$.[1] Let $R \subset M_d(\mathbb{Z}_p) \times M_d(\mathbb{Z}_p)$ be the $\mathbb{Z}_p$-submodule spanned by $\{(\rho(g), \rho'(g)) : g \in G_K\}$. This is a $\mathbb{Z}_p$-algebra, so we can consider the composition
$$\chi\colon G_K \longrightarrow R^\times \twoheadrightarrow (R/p)^\times.$$
This χ is continuous and unramified outside S, and its codomain has cardinality $\leq \#M_d(\mathbb{Z}/p\mathbb{Z})^2 = p^{2d^2}$. Therefore, $\operatorname{im}\chi = \{\chi(\mathrm{Frob}_v) : v \in T\}$. By construction $\operatorname{im}\chi$ spans R/p as an $\mathbb{F}_p$-vector space, so Nakyama implies that
$$\{(\rho(\mathrm{Frob}_v), \rho'(\mathrm{Frob}_v)) : v \in T\}$$
spans R as a $\mathbb{Z}_p$-module. Since taking traces is $\mathbb{Z}_p$-linear, we conclude. ♠♠♠ Niven: [Gesture back at the statement of step 2]
- Step 3: Conclude theorem.
If $\rho\colon G_K \to \mathrm{GL}_d(\mathbb{Q}_p)$ is pure of weight n, then there are only f.many possibilities for the eigenvalues of $\rho(\mathrm{Frob}_v)$. Hence, only f.many possibilities for the function $\operatorname{Tr}\rho\colon G_K \to \mathbb{Q}_p$. This function determines ρ.

■

[1] Follows from the existence of a G_K-stable $\mathbb{Z}_p$-lattice in V. Choose any lattice $\Lambda_0 \subset V$ and consider open subgroup $\mathrm{Aut}_{\mathbb{Z}_p}(\Lambda_0) \subset \mathrm{Aut}_{\mathbb{Q}_p}(V)$. Its preimage under $\rho\colon G \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$ is open and so of finite index, i.e. $G = \bigsqcup_{i=1}^m \gamma_i \cdot \rho^{-1}\left(\mathrm{Aut}_{\mathbb{Z}_p}(\Lambda_0)\right)$. Thus, $\Lambda = \Sigma_{i=1}^m \gamma_i \cdot \Lambda_0$ is a G_K-stable lattice.

Remark 8. In the context of our earlier map $\Psi\colon \mathcal{U}(\mathscr{O}) \to \{\mathbb{Q}_p\text{-reps of } G_K\}$, Lawrence and Venkatesh prove that $\Psi(y)$ is semisimple for all but f.many y. Combined with Faltings' theorem, this implies finiteness of $\operatorname{im}\Psi$. ○

Break here?

Example 9. Say $U = \mathbb{P}^1 - \{0, 1, \infty\}$ and let $X \to U$ be the Legendre family

$$X\colon y^2 = x(x-1)(x-t) \xrightarrow{(x,y,t)\mapsto t} U.$$

Take $n = 1$, so

$$\Psi(t) = T_p X_t \colon G_K \to \mathrm{GL}_2(\mathbb{Q}_p)$$

is the Tate module of the elliptic curve X_t. By Serre's open image theorem, if X_t is not CM, then $\Psi(t)$ is simple. There are only f.many CM j-invariants over K, so X_t will be non-CM for all but f.many t. Hence, $\operatorname{im}\Psi$ is finite for this example. △

2.2. **§2.2: Ψ has finite fibers.** To prove that

$$\begin{array}{rccc} \Psi\colon & \mathcal{U}(\mathscr{O}) & \longrightarrow & \{\mathbb{Q}_p\text{-reps of } G_K\} \\ & y & \longmapsto & \mathrm{H}^n_{\text{ét}}(X_{y,\overline{K}}, \mathbb{Q}_p) \end{array}$$

has finite fibers, we'll use p-adic Hodge theory. Fix a place $v \mid p$ of K as well as a corresponding decomposition group $G_v = G_{K_v} \subset G_K$. It suffices to show the composition

$$\mathcal{U}(\mathscr{O}) \xrightarrow{\Psi} \{\mathbb{Q}_p\text{-reps of } G_K\} \longrightarrow \{\mathbb{Q}_p\text{-reps of } G_v\}$$

has finite fibers. Now, p-adic Hodge theory produces a fully faithful functor

$$D(-)\colon \left\{\begin{array}{c} \mathbb{Q}_p\text{-reps of } G_{K_v} \\ \text{coming from smooth projective schemes}/\mathscr{O}_v \end{array}\right\} \longrightarrow \left\{\begin{array}{c} \text{filtered vector spaces}/K_v \\ \text{w/ a semilinear Frobenius endomorphism} \end{array}\right\} =: \mathrm{MF}^{\varphi}_{K_v}$$

for which

$$D\Big(\mathrm{H}^n_{\text{ét}}(X_{y,\overline{K}_v}, \mathbb{Q}_p)\Big) \simeq \mathrm{H}^n_{\mathrm{dR}}(X_y/K_v) \text{ w/ Hodge filtration and some Frobenius.}$$

Fact ("Fact"). As $y \in \mathcal{U}(\mathscr{O})$ varies, $\mathrm{H}^n_{\mathrm{dR}}(X_y/K_v) = \mathrm{H}^n_{\mathrm{dR}}(X_y/K) \otimes_K K_v$ and the Frobenius stay the same, but the Hodge filtration changes. *{□ This is imprecise as stated, but it means that we can restrict Ψ to a map which keeps track of the variation of the Hodge filtration of $\mathrm{H}^n_{\mathrm{dR}}$ w/o losing much information. □}*

Notation 10. Fix a point $y_0 \in \mathcal{U}(\mathscr{O})$ and set $X_0 := X_{y_0}$. Let $\mathcal{H}/K$ be the K-variety parameterizing flags in $\mathrm{H}^n_{\mathrm{dR}}(X_0/K) =: V_0$ of the same shape as its Hodge filtration.

Fact. The relative de Rham cohomology of X/U over K supports a Gauss–Manin connection. For any point y in the residue disk $]y_0[:= \{y \in \mathcal{U}(\mathscr{O}_v) : y \equiv y_0 \mod v\}$, this gives rise to a Frobenius-equivariant isomorphism

$$\mathrm{GM}\colon \mathrm{H}^n_{\mathrm{dR}}(X_y/K_v) \xrightarrow{\sim} \mathrm{H}^n_{\mathrm{dR}}(X_0/K_v) \simeq V_0 \otimes_K K_v,$$

and so to a K_v-analytic period map $\Phi\colon]y_0[\longrightarrow \mathcal{H}(K_v)$ sitting in a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{U}(\mathscr{O}) & \hookrightarrow & \mathcal{U}(\mathscr{O}_v) & \hookleftarrow &]y_0[& & \\
\downarrow{\scriptstyle\Psi} & & \downarrow & & \downarrow{\scriptstyle \mathrm{H}^n_{\mathrm{dR}} + \mathrm{GM}} & \searrow{\scriptstyle\Phi} & \\
\left\{\begin{array}{c}\text{“nice”}\\ \mathbb{Q}_p\text{-reps of } G_K\end{array}\right\} & \longrightarrow & \left\{\begin{array}{c}\text{“nice”}\\ \mathbb{Q}_p\text{-reps of } G_v\end{array}\right\} & & & & \\
 & & \downarrow{\scriptstyle D} & & & & \\
 & & \mathrm{MF}^{\varphi}_{K_v} & \xleftarrow[\text{forget}]{} & \left\{\Big(V \in \mathrm{MF}^{\varphi}_{K_v}, V \xrightarrow{\sim} \mathrm{H}^n_{\mathrm{dR}}(X_0/K_v)\Big)\right\} & \longrightarrow & \mathcal{H}(K_v).
\end{array}$$

Remark 11. Let $\mathrm{Aut}(V_0, \mathrm{Frob}) \subset \mathrm{GL}(V_0)$ be the set of automorphisms commuting with (the $[K_v : \mathbb{Q}_p]$th power of) Frobenius on V_0. Then, fibers of the "forget" map are $\mathrm{Aut}(V_0, \mathrm{Frob})$-orbits. Thus, Ψ will have finite fibers if (every choice of y_0) the Φ-preimage of any $\mathrm{Aut}(V_0, \mathrm{Frob})$-orbit in $\mathcal{H}(K_v)$ is finite. ○

Corollary 12. *If* $\dim \mathrm{Aut}(V_0, \mathrm{Frob}) < \dim \overline{\Phi(]y_0[)}$*, then* Ψ *has finite fibers.*

♠♠♠ Niven: [Talk ended up only getting to statement of this corollary. ☹]

Proof. Since Ψ has finite image, $\Phi(]y_0[\cap \mathcal{U}(\mathscr{O})) \subset \mathcal{H}(K_v)$ is contained in f.many $\mathrm{Aut}(V_0, \mathrm{Frob})$-orbits. Hence, there will be some analytic function on $]y_0[$ vanishing on $\mathcal{U}(\mathscr{O}) \cap]y_0[\subset \mathcal{U}(\mathscr{O}_v)$ ■

{□ *At this point, we're almost done. We need a way of estimating the dimension of the (Zariski) closure of the* v*-adic period map. This will come from a comparison with a complex analogue of the period map.* □}

Remark 13. Because the Gauss–Manin connection is defined over K, it also gives a way of identifying the cohomology of nearby fibers over $\mathbb{C}$, i.e. it gives rise to a connection (and so to a local system) on de Rham cohomology/$\mathbb{C}$. ○

Remark 14. Let $\widetilde{U}$ be the universal cover of $U(\mathbb{C})$, and let $\widetilde{X} = X \times_{U(\mathbb{C})} \widetilde{U}$. Then, for any $y \in \widetilde{U}$, we get an isomorphism

$$\mathrm{GM}_{\mathbb{C}} \colon \mathrm{H}^n_{\mathrm{dR}}(\widetilde{X}_y/\mathbb{C}) \longrightarrow \mathrm{H}^n_{\mathrm{dR}}(X_0/\mathbb{C}) \simeq V_0 \otimes_K \mathbb{C},$$

and so we get a complex analytic period map $\Phi_{\mathbb{C}} \colon \widetilde{U} \to \mathcal{H}(\mathbb{C})$. ○

Fact.

- The Gauss–Manin connection induces a monodromy action

$$\pi_1(U(\mathbb{C}), y_0) \longrightarrow \mathrm{Aut}\, \mathrm{H}^n_{\mathrm{dR}}(X_0/\mathbb{C}),$$

  and $\Phi_{\mathbb{C}}$ is equivariant w.r.t the induced action on $\mathcal{H}(\mathbb{C})$.
- Because the Gauss–Manin connection is defined over K, the Zariski closures of the images of Φ and $\Phi_{\mathbb{C}}$ have the same dimension; in fact, there is a single subscheme $Z \subset \mathcal{H}$ defined over K which realizes both closures!

Corollary 15. *Let* $h \in \mathcal{H}(K)$ *be the point corresponding to the Hodge filtration on* $\mathrm{H}^n_{\mathrm{dR}}(X_0/K) = V_0$*. Then,* $\dim \overline{\Phi(]y_0[)} \geq \dim \overline{\pi_1(U(\mathbb{C}), y_0) \cdot h}$*. In particular,*

$$\dim \mathrm{Aut}(V, \mathrm{Frob}) < \dim \overline{\pi_1(U(\mathbb{C}), y_0) \cdot h} \implies \Psi \text{ has finite fibers.}$$

Final remarks

- To conclude, one wants to prove a "large monodromy" theorem saying that the RHS is big.
  For this, it helps if the fibers of $X \to U$ are <u>disconnected</u>, as this will make $\mathrm{Aut}\, \mathrm{H}^n_{\mathrm{dR}}(X_0/\mathbb{C})$ bigger.
- One should also hope that the Frobenius centralizer $\mathrm{Aut}(V, \mathrm{Frob})$ is small.
  For this, it helps to enlarge K_v (so it's hard to commute with Frobenius).
- We'll see this strategy played out twice, once for $\mathbb{P}^1 - \{0, 1, \infty\}$ and once to prove Mordell in full. L–V give a third application in their paper, for studying degree d hypersurfaces with good reduction outside S.
  ♠♠♠ Niven: [Integral points on the moduli space of smooth hypersurfaces of degree d in $\mathbb{P}^n$ are not Zariski dense]

3. Overfill (Stealing 15 minutes from next talk)

Recall 16. Have smooth proper family $\pi\colon X \to U$ over K extending to a smooth, projective $\pi\colon \mathcal{X} \to \mathcal{U}$ over $\mathscr{O}$. We fixed $y_0 \in \mathcal{U}(\mathscr{O})$, let $V_0 \coloneqq \mathrm{H}^n_{\mathrm{dR}}(X_0/K)$, and let $\mathcal{H}/K$ be the flag variety parameterizing "Hodge-type" filtrations of V_0. Big diagram:

$$\begin{array}{ccccccccc}
y & \in & \mathcal{U}(\mathscr{O}) & \hookrightarrow & \mathcal{U}(\mathscr{O}_v) & \hookleftarrow &]y_0[& & \\
\downarrow\!\!\!\!\mapsto & & \downarrow \Psi & & \downarrow & & \downarrow \mathrm{H}^n_{\mathrm{dR}} + \mathrm{GM} & \searrow \Phi & \\
\mathrm{H}^n_{\text{ét}}(X_{y,\overline{K}}, \mathbb{Q}_p) & \in & \left\{ \begin{matrix} \text{"nice"} \\ \mathbb{Q}_p\text{-reps of } G_K \end{matrix} \right\} & \longrightarrow & \left\{ \begin{matrix} \text{"nice"} \\ \mathbb{Q}_p\text{-reps of } G_v \end{matrix} \right\} & & & & \\
 & & & & \downarrow D & & & & \\
 & & & & \mathrm{MF}^{\varphi}_{K_v} & \xleftarrow[\text{forget}]{} & \left\{ \left(V \in \mathrm{MF}^{\varphi}_{K_v}, V \xrightarrow{\sim} V_0 \otimes_K K_v \right) \right\} & \longrightarrow & \mathcal{H}(K_v).
\end{array}$$

Fibers of "forget" are $\mathrm{Aut}(V_0, \mathrm{Frob})$-orbits. ⊙

Corollary 17. *If* $\dim \mathrm{Aut}(V_0, \mathrm{Frob}) < \dim \overline{\Phi(]y_0[)}$, *then* Ψ *has finite fibers.*

♠♠♠ Niven: [Everything above on board before starting.] Jump to the proof of Corollary 12...