Assumption: All schemes locally noetherian & all morphisms lft

## Brief Review of Flatness

Defn: $A \to B$ is flat if $B \otimes_A -$ is exact

Prop: $A \xrightarrow{\varphi} B$ is flat iff

$A_{\varphi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$ is flat $\forall \mathfrak{p} \subset \operatorname{spec} B$

Pf: $\Rightarrow$:

$$A_{\varphi^{-1}(\mathfrak{p})} \xrightarrow{\text{flat}} B_{\varphi^{-1}(\mathfrak{p})} \xrightarrow{\text{flat}} B_{\mathfrak{p}}$$

$\Leftarrow$: $B \otimes_A -$ exact

$\Leftrightarrow B_{\mathfrak{p}} \otimes_A -$ exact $\forall \mathfrak{p}$

$$A \xrightarrow{\text{flat}} A_{\varphi^{-1}(\mathfrak{p})} \xrightarrow{\text{flat}} B_{\mathfrak{p}}$$

**Thm/defn**: A morphism $X \xrightarrow{f} Y$ is flat if either holds

(i) $\forall$ affine opens $\operatorname{spec} A \subset Y$ & $\operatorname{spec} B \subset f^{-1}(\operatorname{spec} A)$ the induced $A \to B$ is flat.

(ii) The maps $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are flat $\forall x \in X$

**Thm**: For $X \xrightarrow{f} Y$ flat, then

$$\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{X_y,x}$$

where $y = f(x)$

# Regular Schemes

<u>Defn</u>: Given a local ring

$(A, \mathfrak{m}, \kappa)$

max — residue field

we say it is <u>regular</u> if

$$\dim_\kappa \underbrace{\mathfrak{m}/\mathfrak{m}^2}_{\text{Zariski cotangent}} = \dim A$$

<u>Defn</u>: A scheme $X$ is <u>regular</u> if all local rings $\mathcal{O}_{X,x}$ are regular local rings. It suffices to check that $\mathcal{O}_{X,x}$ is regular when $x$ is closed.

Ex. $\mathbb{A}^n_k$ is regular

at $O \in \mathbb{A}^n_k$, we have

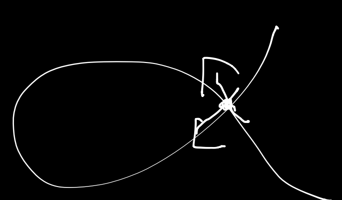
$$\frac{\mathfrak{m}}{\mathfrak{m}^2} = \frac{(x_1, \dots, x_n)}{(x_i x_j, x_i^2)}$$

has dimensional $n$

(shows regularity @ origin)

Ex. $\mathbb{P}^1_k$ is regular

Non-ex. $X = \{y^2 = x^3 + x^2\}$



2-dim tangent space on 1-dim curve

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} = \frac{(x,y)}{(x^2, xy, y^2, y^2 - x^3 - x^2)}$$

$$= \frac{(x,y)}{(x^2, xy, y^2)}$$

is 2-dimensional

Fact: $\operatorname{Spec} \underbrace{\frac{K[t_1, \ldots, t_n]}{(f_1, \ldots, f_m)}}_{=: A} =: X$

$$\begin{array}{c} \downarrow \\ \operatorname{Spec} K \end{array}$$

$f_1, \ldots, f_m$ reg

$f_1$ not a ZD in $A$

$f_2$ not a ZD $A/(f_1)$

$f_3$ not a ZD $A/(f_1, f_2)$

Assume $f_1, \ldots, f_m$ regular seq

so $\dim X = n - m$

At a closed pt $x \in X(k)$, one has the Jacobian

$$\left(\frac{\partial f_i}{\partial t_j}\right)(x) = J_f(x)$$

$$\operatorname{Coker} J_f(x) \cong \mathfrak{m}_x/\mathfrak{m}_x^2$$

so $X/k$ regular at $x$

$$\Longleftrightarrow \operatorname{corank} J_f(x) = n - m$$

$$\Longleftrightarrow \operatorname{rank} \underset{\substack{\uparrow \\ m \times n \text{ matrix}}}{J_f(x)} = m$$

$\Gamma$

$$\operatorname{Coker} J_f(x) \cong \mathfrak{m}/\mathfrak{m}^2$$

$$
\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & \mathbb{A}^n \arrow[d, "f=(f_1,\dots,f_n)"] \\
0 \arrow[r, hook] & \mathbb{A}^m
\end{tikzcd}
$$

$$T_x\mathbb{A}^n \xrightarrow{J_f(x)} T_x\mathbb{A}^m$$

<u>Defn:</u> A scheme $X/k$ is <u>geometrically regular</u> if $X_{\bar{k}} = X \times_k \bar{k}$ is regular.

*Defn:* A scheme $X$ is *geometrically regular at* $\underline{x} \in X$ if for all $x' \in X_{\overline{k(x)}}$ above $x$, $X_{\overline{k(x)}}$ is regular at $x'$.

($\iff$ $(X_{\overline{k(x)}})_{x}$ is regular)

## Smooth morphisms

Thm/defn: $X \xrightarrow{f} Y$ is smooth of relative dimension $r$ at $x \in X$ if either

(i) $f$ is flat, $X_y$ is geometrically regular at $x$ $(y = f(x))$, and $\dim_x X_y = r$.

(ii) $\exists$ affine opens $\underset{f(x)=y}{V \ni} \subset Y$ & $\underset{x}{U \ni} \subset f^{-1}(V)$

along w/ a commutative diag

$$\begin{tikzcd} U \arrow[r, "\simeq"] \arrow[d, "f"'] & \operatorname{Spec} \dfrac{A[t_1, \ldots, t_n]}{(f_1, \ldots, f_{n-r})} \arrow[d] \\ V \arrow[r, "\simeq"] & \operatorname{Spec} A \end{tikzcd}$$

rel dim $r$

such that the Jacobian

$$\underbrace{\left(\frac{\partial f_i}{\partial t_j}\right)(x)}$$

$(n-r) \times n$ matrix w/ coeff in $k(x)$

has rank $n-r$.

Pf: [Bosch et al., "Néron models"]

[§2.4 Prop 8 + §2.2 Prop 15]

**Defn:** The smooth locus of $X \xrightarrow{f} Y$ is

$$X^{sm} = \left\{ x \in X : f \text{ is smooth } @\, x \right\}$$

**Prop:** $X^{sm} \subset X$ is open.

**Cor:** If $X/K$ is geometrically reduced, then $X^{sm} \subset X$ dense.

Non-ex: $X = \{ x^p - t y^p = 0 \}$ over $\mathbb{F}_p(t)$

$X$ reduced but not geom reduced

since $t^{1/p} \in \overline{\mathbb{F}_p(t)}$

& $(x^p - ty^p) = (x - t^{1/p}y)^p$

Can see $X^{sm} = \emptyset$

Prop: If $X \longrightarrow Y$ smooth of rel. dim $r$, then $\Omega_{X/Y}$ is locally free of rank $r$ (on $X$).

Pf idea: $\operatorname{Spec} A[t_1, \ldots t_n]/(f_1, \ldots f_m)$

$$\downarrow$$

$\operatorname{Spec} A$

$\Omega_{X/A}$ generated by

$dt_1, \dots, dt_n$ subject to

$$0 = df_j = \sum \frac{\partial f_j}{\partial t_i} dt_i$$

these relations

$$J_f(x) \begin{pmatrix} dt_1 \\ \vdots \\ dt_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Rmk. Another characterization of smoothness. Turns out $X \to Y$ is smooth iff

$\forall$ $\operatorname{spec} A \to Y$ and any nilpotent ideal $I \subset A$, the

map $X(A) \to X(A/I)$
$(X(A) = \mathrm{Hom}_k(\mathrm{spec}\,A, X))$
is surjective.

## Étale morphisms

**Defn** $X \to Y$ <u>étale at $x$</u> if it smooth of rel dim 0 at $x$.

**Prop**: if $X/K$ étale, then

$$X = \coprod_{i \in I} \mathrm{spec}\, L_i$$

where $L_i/K$ finite, separable.

**Defn:** A map $X \to Y$ is unramified at $x$ if

$$\mathfrak{m}_{X,x} = \mathfrak{m}_{Y,f(x)} \mathcal{O}_{X,x}$$

$$\left( \Leftrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_{Y,f(x)} = k(x) \right)$$

+ $k(x)/k(y)$ separable.

**Thm:** $X \to Y$ étale

$\Leftrightarrow$ flat + unramified

**Ex.** $L/K$ number fields

$\operatorname{Spec} \mathcal{O}_L$

$$\downarrow$$

$$\operatorname{Spec} \mathcal{O}_K$$

is étale at $\mathfrak{P} \in \operatorname{Spec} \mathcal{O}_L$ iff its unramified in the algebraic theory sense.

Ex. open inversions are étale

Ex. $\mathbb{G}_m \xrightarrow{[n]} \mathbb{G}_m$

$\operatorname{Spec} k[x, x^{-1}] \; \lrcorner \qquad x \longmapsto x^n$

is étale $\Longleftrightarrow \operatorname{char} k \nmid n$

$\mathbb{G}_m(\mathbb{C}) \simeq \mathbb{C}^\times \quad \frac{\mathbb{Z}}{}$

$$\mathbb{C}^\times \quad \mathbb{Z}^n$$

**Prop:** For $X \xrightarrow{f} Y$ étale of finite type, we have ($y = f(x)$)

(i) $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y}$

(ii) $f$ is quasi-finite

(iii) The map on tangent spaces

$$T_x X \longrightarrow T_y Y \otimes k(x)$$

is an isomorphism.