The Speeds of Families of Intersection Graphs

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Abstract

A fundamental question of graph theory lies in counting the number of graphs which satisfy certain properties. In particular, the structure of intersection graphs of planar curves is unknown, and Schaefer, Sedgwick, and Štefankovič have shown that recognizing such graphs is NP-complete. As such, the number of intersection graphs of planar curves poses an interesting problem.

Pach and Tóth have previously proven bounds on the number of intersection graphs of string graphs. We investigate the more specific case of the number of intersection graphs on n vertices of systems of segments of certain algebraic curves, including parabolas, conic sections, polynomials, and rational functions. We extend the results of Pach and Solymosi, who obtained upper bounds on the number of intersection graphs of line segments, and Fox, who obtained tight lower bounds.

For each system we establish a set of polynomials whose sign patterns give an intersection graph. We use Warren's Theorem to obtain an upper bound on the number of sign patterns of this set. We then use a constructive approach to calculate matching lower bounds on the number of intersection graphs. In general, the bounds on the intersection graphs of these systems is $n^{n(f+o(1))}$, where f is the degree of freedom.

1 Introduction

A fundamental question of graph theory lies in counting the number of graphs which satisfy certain properties, such as the number of cyclic graphs, connected graphs, and bipartite graphs.

The *speed* $s_F(n)$ of a family of graphs $\mathscr{F}$ is defined by the number of graphs inside the family with n vertices. We are interested in the speeds of *hereditary families* of graphs, i.e. graphs such that their induced subgraphs are also in the family. An example of these would be planar graphs. O. Giménez and M. Noy [4] have previously proven that the precise asymptotic estimate of the speed of the family of planar graphs is $s_F(n) \sim gn^{-\frac{7}{2}}\gamma^n n!$ where g and γ are both constants. Note that we are dealing with labeled graphs, so each vertex is distinct. Two graphs are isomorphic and considered to be the same graph if they contain the same number of vertices connected by edges in the same way.

In this paper, we are interested more specifically in the *intersection graph* of a family of sets. Given a finite family of sets $\{S_i\}$, the intersection graph of $\{S_i\}$ is a graph such that every set S_i is a vertex and every pair of vertices (v_i, v_j) is connected by an edge if and only if S_i and S_j intersect. O. Giménez and M. Noy's result is related to intersection graphs through the Koebe-Andreev-Thurston circle packing theorem [5], which states that there exists an intersection graph of a circle packing in a plane isomorphic to every planar graph, where a circle packing is an arrangement of non-intersecting circles, some of which must be mutually tangent.

In particular, the speed of intersection graphs of planar curves poses an interesting problem since the structure of such graphs is unknown. Moreover, determining whether any given graph is an intersection graph of planar curves is NP-complete; J. Kratochvíl [6] proved this question to be NP-hard, and M. Schaefer, E. Sedgwick, D. Štefankovič [10] proved this question to be NP, thus making the problem NP-complete.

The general problem of finding the speed of certain families of intersection graphs is elusive. However, in some special cases, we can find bounds for this number. J. Pach and J. Solymosi [8] obtained an upper bound of $n^{n(4+o(1))}$ for the speed of intersection graphs of line segments in the plane $\mathbb{R}^2$, using Warren's Theorem [13]. Given that each unique combination of signs of a system of polynomials, considering any range of values for the variables, is a sign pattern, Warren's Theorem states that the number of different sign patterns in m polynomials with degree $d \geq 1$ and in v variables is at most $(4edm/v)^v$, where e is Euler's constant. Note that we are dealing with polynomials over the reals throughout this paper. Warren's Theorem is closely related with *degrees of freedom*. Colloquially, the degrees of freedom are the number of variables needed to define each segment. J. Spinrad [11] gave a full explanation and discussion of Warren's Theorem. N. Alon and E. Scheinerman [1] used Warren's Theorem and degrees of freedom to count the classes of partial orders. We use a similar idea here, to obtain the upper bounds on the speeds of intersection graphs of various families. J. Pach posed the question of finding the speeds of intersection graphs of other algebraic curves, which we discuss for certain cases in this paper. J. Fox [3] found the asymptotic lower bounds for the speeds of intersection graphs of families of line segments to be $n^{n(4-o(1))}$, thus showing constructively that the exponent $4n$ is correct.

The speeds of intersection graphs have been studied extensively, generating various related open problems. In particular, J. Pach and G. Tóth [9] have proven, through a constructive argument, the bounds for the more general intersection graphs of curves in the plane, or string graphs. They obtained an upper bound of $2^{\left(\frac{3}{4}+o(1)\right)\binom{n}{2}}$ and a lower bound of $2^{\frac{3}{4}\binom{n}{2}}$ for string graphs on n unlabeled vertices. J. Kynčl [7] generalized their results and obtained an upper bound of $2^{O\left(n^{3/2}\log n\right)}$ on the number of intersection graphs of pseudosegments, or strings with exactly one point of intersection.

Furthermore, we are also interested in the properties of the intersection graphs that can be created with other functions such as conic sections. G. Ehrlich, S. Even, and R. E. Tarjan [2] have constructed graphs which are not intersection graphs and intersection graphs which are not intersection graphs of straight lines.

Our main goal is to extend these results first to the more specific case of families of segments of conic sections and polynomials, and then to segments of planes in $\mathbb{R}^3$, where segments simply refer to a constrained portion of each function, such as in line segments. In Section 2, we obtain the upper bounds on the number of intersection graphs of segments of parabolas, rotated parabolas, conics sections, planes in $\mathbb{R}^3$, polynomials, and rational functions. Each of these results follow a more general result, namely:

Theorem 1. *The number of intersection graphs of n segments is at most $n^{n(f+o(1))}$ where f is the degree of freedom of the system of segments.*

In Section 3, we obtain essentially tight lower bounds on the number of intersection graphs of segments of parabolas, rotated parabolas, polynomials, and rational functions.

2 Upper Bounds on the Number of Intersection Graphs

In this section, we derive upper bounds for various families of intersection graphs, starting with parabolas and using a similar construction for conic sections and planes in $\mathbb{R}^3$. We then extend our results to polynomials and rational functions, through a slightly different approach.

2.1 Preliminaries and Definitions

In this subsection, we introduce key notation that will be used throughout the section.

Definition 1. A *segment* of a function refers to the image of that function as bounded by certain *constraints* — for example, the constraint $\ell_i \leq x \leq u_i$ on a function $y = f(x)$ defines a segment. We denote segments by capital letters, such as A_i. An *unconstrained* segment refers to the function $y = f(x)$ of the segment, without taking the bounds into consideration.

For any variables a_i and a_j, denote the difference $a_i - a_j$ as a_{ij}.

For any segments A_i and A_j where we write the unconstrained A_i in the form $y = f(x)$ and the unconstrained A_j in the form $y = g(x)$, we define A_{ij} to be the difference $f(x) - g(x)$. Note that the roots of A_{ij} give the x-coordinate of the intersection between the unconstrained A_i and A_j.

Definition 2. Given that the bounds of A_i are $\ell_i \leq x \leq u_i$ and the bounds of A_j are $\ell_j \leq x \leq u_j$, we say the roots of A_{ij} are *within the constraints* if they are bounded between $\max\{\ell_i, \ell_j\}$ and $\min\{u_i, u_j\}$.

Note that if the roots of A_{ij} are within the constraints, then the segments A_i and A_j intersect.

Furthermore, note that for functions in n dimensions, we can naturally extend this definition. Let a segment B_i represent a function of variables $b_1, b_2, \ldots, b_n$ with some constraints $\ell_{k_i} \leq b_k \leq u_{k_i}$ for each $k = 1, 2, \ldots, n-1$. For any segments B_i and B_j where we write the unconstrained B_i in the form $b_n = f(b_1, b_2, \ldots, b_{n-1})$ and the unconstrained B_j in the form $b_n = g(b_1, b_2, \ldots, b_{n-1})$, we define B_{ij} to be the difference $f(b_1, b_2, \ldots, b_{n-1}) - g(b_1, b_2, \ldots, b_{n-1})$. The roots of B_{ij} follow analogously to the 2 dimensional case.

Definition 3. The roots of B_{ij} are said to be *within the constraints* if each of the roots b_k for $k = 1, 2, \ldots, n-1$ are bounded by the corresponding $\max\{\ell_{k_i}, \ell_{k_j}\}$ and $\min\{u_{k_i}, u_{k_j}\}$.

We also use the following definition of sign patterns of a system of polynomials. Let $\mathbf{p} = (p_1(\mathbf{x}), \ldots, p_m(\mathbf{x}))$ be a sequence of m polynomials in the v variables $\mathbf{x} \in \mathbb{R}^v$.

Definition 4. The number of *sign patterns* of $\mathbf{p}$ is the number of different vectors $(sgn(p_1(\mathbf{x})), \ldots, sgn(p_m(\mathbf{x}))) \in \{-1, 0, +1\}^m$ considering all possible values of $\mathbf{x}$.

For example, considering the system of polynomials $p_1(a,b) = a$ and $p_2(a,b) = b$, we have 2 polynomials of maximum degree 1 utilizing 2 variables, namely a and b. Since a can be positive or negative and b can be positive or negative, we have 4 different possible sign patterns.

We then use the following form of the Milnor-Thom Theorem, known as Warren's Theorem [13].

Warren's Theorem. *If $m \geq v$, the number of different sign patterns for m polynomials of degree at most d in v variables is at most $(4edm/v)^v$, where e is Euler's constant.*

Note that Warren's Theorem does not account for the case where a value is zero. N. Alon and E. Scheinerman [1] rectify this by 'doubling' each polynomial, giving an upper bound of $(8edm/v)^v$. This merely changes the coefficient of the bound, and as such is insignificant when considering the asymptotic bounds in this section.

For a more general result, we consider the *degrees of freedom* of a family of segments $\mathscr{A}$, as inspired by N. Alon and E. Scheinerman's [1] definition of degrees of freedom.

Definition 5. The segments in $\mathscr{A}$ have f *degrees of freedom* if

1. There is an injection $\phi\colon \mathscr{A} \to \mathbb{R}^f$.
2. There exist polynomials $p_1, p_2, \ldots, p_m \in \mathbb{R}[x_1, \ldots, x_f]$ such that for all $S, T \in \mathscr{A}$, whether $S \cap T \neq \emptyset$ can be determined based on the signs of all possible values of these polynomials.

The bounds on the number of intersection graphs of the systems we investigate in this paper can be written in terms of degrees of freedom, as stated in Theorem 1.

2.2 Upper Bounds on Parabolas

We begin by considering the upper bounds on the number of intersection graphs of segments of parabolas with vertical axes of symmetry.

Let $\mathscr{Q} = \{Q_1, Q_2, \ldots, Q_n\}$ be a system of segments of parabolas in the plane. Assume that Q_i is not degenerate and can be described by the relations

$$Q_i : y = a_i x^2 + b_i x + c_i \text{ where } \ell_i \leq x \leq u_i \text{ for } i = 1, 2, \ldots, n.$$

Theorem 2. *The number of intersection graphs of n segments of parabolas is $n^{n(5+o(1))}$.*

Proof. Two segments, Q_i and Q_j, cross each other if and only if the roots of $Q_{ij} = a_{ij}x^2 + b_{ij}x + c_{ij}$ are within the constraints. So, they cross if and only if at least one of the following inequalities is satisfied

$$\max\{\ell_i, \ell_j\} \leq \frac{-b_{ij} + \sqrt{b_{ij}^2 - 4a_{ij}c_{ij}}}{2a_{ij}} \leq \min\{u_i, u_j\}, \tag{1}$$

$$\max\{\ell_i, \ell_j\} \leq \frac{-b_{ij} - \sqrt{b_{ij}^2 - 4a_{ij}c_{ij}}}{2a_{ij}} \leq \min\{u_i, u_j\}. \tag{2}$$

Using these inequalities, we can construct a system of polynomials where the unique sign patterns determine whether or not Q_i and Q_j intersect within the given constraints. From this system, we can apply Warren's Theorem to obtain an upper bound on the number of possible different crossings, and thus an upper bound on the number of intersection graphs.

We begin by simplifying the above inequalities; our goal is to receive an equivalent system of inequalities of the form $p(x) \leq 0$ or $p(x) \geq 0$ where $p(x)$ is a polynomial. As such, the sign patterns of the system of polynomials $p(x)$ will indicate whether or not Q_i and Q_j cross.

We first consider $b_{ij}^2 - 4a_{ij}c_{ij}$ as one such polynomial in the system. If the sign is negative, the unconstrained Q_i and Q_j will not cross at all. If it is positive, then we can continue with simplifying the inequality.

We multiply all sides of both Equations (1) and (2) by a_{ij}, but this operation can change the $\leq$ signs to $\geq$ signs if a_{ij} is negative. As such, a_{ij} is another polynomial we must add to the system. We can continue with simplifying, through casework, considering all four of the constraints.

However, we can also view these inequalities graphically, giving us fewer polynomials in the final system. If there are real roots of Q_{ij}, as based on the discriminant, then it is sufficient to know the location of the constraints with respect to the vertex of Q_{ij}. Consider each of the constraints $t = \ell_i, \ell_j, u_i, u_j$. For each of the constraints, the sign of $a_{ij}t^2 + b_{ij}t + c_{ij}$ determines if the constraints are above or below the vertex of Q_{ij}, and as such if the constraints contain the vertex vertically. This is because we know the signs of a_{ij} and the discriminant, so we know whether the ordinate of the vertex of Q_{ij} is positive or negative. The sign of $2ta_{ij} - b_{ij}$ determines if the constraints are to the left or the right of the vertex of Q_{ij}, and so if the constraints contain the vertex horizontally.

In these inequalities, we assume that $a_{ij} \neq 0$ and that the unconstrained Q_i and Q_j intersect twice. If the unconstrained Q_i and Q_j intersect exactly once, then $a_{ij} = 0$ and $Q_{ij} = b_{ij}x + c_{ij}$. The constrained Q_i and Q_j cross each other if and only if the roots of Q_{ij} are within the constraints. They cross each other if and only if

$$\max\{\ell_i, \ell_j\} \leq \frac{-c_{ij}}{b_{ij}} \leq \min\{u_i, u_j\}.$$

We add more polynomials to the system in order to account for this case. Again, we begin by simplifying the inequality. Analogous to the previous case, we must add b_{ij} and $tb_{ij} + c_{ij}$ for all $t = \ell_i$, ℓ_j, u_i, and u_j to the system.

Combining all of the above cases of possible crossings of Q_i and Q_j, the system of polynomials becomes:

$$\begin{cases} b_{ij}^2 - 4a_{ij}c_{ij} \\ a_{ij} \\ a_{ij}t^2 + b_{ij}t + c_{ij} \ \text{ for } \ t = \ell_i, \ell_j, u_i, u_j \\ 2ta_{ij} - b_{ij} \ \text{ for } \ t = \ell_i, \ell_j, u_i, u_j \\ b_{ij} \\ tb_{ij} + c_{ij} \ \text{ for } \ t = \ell_i, \ell_j, u_i, u_j \end{cases}$$

for any distinct i and j from $\{1, 2, \ldots, n\}$. Applying Warren's Theorem, we obtain an upper bound on the number of intersection graphs of segments of parabolas, based on the number of sign patterns of the above system. We have 15 polynomials for each pair i, j, so $15\binom{n}{2}$ polynomials in total. The degree of the system is at most 3, and the system uses $5n$ variables. We obtain that there are at most $n^{n(5+o(1))}$ graphs on n vertices which are intersection graphs of segments of parabolas. $\square$

2.3 Upper Bounds on Rotated Parabolas and Conic Sections

We now extend the construction for the upper bounds on the number of intersection graphs of segments of parabolas to segments of conic sections, and by a simple constraint, to segments of rotated parabolas.

Let $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ be a system of segments of conic sections in the plane. Assume that S_i is not degenerate and can be described by the relations

$$S_i : a_i x^2 + y^2 + c_i xy + d_i x + e_i y + f_i = 0 \text{ where } \ell_i \leq x \leq u_i \text{ for } i = 1, 2, \ldots, n.$$

Theorem 3. *The number of intersection graphs of segments of conic sections is $n^{n(7+o(1))}$.*

Proof. Two segments, S_i and S_j, cross each other if and only if the roots of S_{ij} are within the constraints. We can find the roots of S_{ij} using the quartic equation, and applying the constraints $\max\{\ell_i, \ell_j\} \leq x \leq \min\{u_i, u_j\}$, we receive a larger system of polynomials whose signs determine whether two segments intersect. Note that creating such a system of polynomials is possible since the inequalities are based on the quartic equation, so they involve some combination of fractional and integer exponents. These can be evaluated to create a finite number of polynomial inequalities, with a certain amount of casework.

So, we have, for some constant k, $k\binom{n}{2}$ polynomials of degree at most 7 in $7n$ variables. Applying Warren's Theorem, we obtain that there are at most $n^{n(7+o(1))}$ intersection graphs of segments of conic sections.

Note that the equation for a rotated parabola is the same as that for a general conic, except with the stipulation that $4a_ic_i = 1$ for all $i = 1,2,\ldots,n$. As such, only $6n$ variables are necessary to construct, for some constant k, $k\binom{n}{2}$ polynomials of degree at most 7. Applying Warren's Theorem, we obtain that there are at most $n^{n(6+o(1))}$ graphs on n vertices which are intersection graphs of segments of rotated parabolas. $\square$

2.4 Upper Bounds on Planes

We consider the upper bounds on the number of intersection graphs of segments of planes.

Let $\mathscr{R} = \{R_1, R_2, \ldots, R_n\}$ be a system of segments of planes in $\mathbb{R}^3$. Assume that R_i is not parallel to the xy-plane. Since there is a finite number of planes, we can merely rotate the system to receive the same intersection graph but without a plane parallel to the xy-plane. Let R_i be described by the relations

$$R_i : z = a_ix + b_iy + d_i \text{ where } \ell_{x_i} \leq x \leq u_{x_i} \text{ and } \ell_{y_i} \leq y \leq u_{y_i} \text{ for } i = 1,2,\ldots,n.$$

Theorem 4. *The number of intersection graphs of n segments of planes is $n^{n(7+o(1))}$.*

Proof. Two segments, R_i and R_j, cross each other if and only if the roots of $R_{ij} = a_{ij}x + b_{ij}y + d_{ij}$ lie within the constraints. So, they cross if and only if

$$\max\{\ell_{y_i}, \ell_{y_j}\} \leq -\frac{d_{ij} + xa_{ij}}{b_{ij}} \leq \min\{u_{y_i}, u_{y_j}\}$$

and

$$\max\{\ell_{x_i}, \ell_{x_j}\} \leq -\frac{d_{ij} + yb_{ij}}{a_{ij}} \leq \min\{u_{x_i}, u_{x_j}\}.$$

Using these inequalities, we can construct a system of polynomials where the unique sign patterns determine whether or not R_i and R_j intersect within the given constraints. It is sufficient to first consider the top inequality and simplify it considering only the ℓ_{y_i} constraint for now to obtain

$$\frac{\ell_{y_i} b_{ij} + d_{ij}}{-a_i j} \leq x.$$

We can then apply the constraints on x, to get

$$\max\{\ell_{x_i}, \ell_{x_j}\} \leq \frac{\ell_{y_i} b_{ij} + d_{ij}}{-a_i j} \leq \min\{u_{x_i}, u_{x_j}\}.$$

We can simplify this inequality in the same manner as before. Note that we must add a_{ij} and b_{ij} to the system, as we multiply all sides by both and the operation can change the $\leq$ signs to $\geq$ signs. We can continue simplifying through casework for each of the constraints. Considering all $t_x = \ell_{x_i}, \ell_{x_j}, u_{x_i}, u_{x_j}$ and $t_y = \ell_{y_i}, \ell_{y_j}, u_{y_i}, u_{y_j}$, we see that we must consider the sign of $t_y b_{ij} + d_{ij} + t_x a_{ij}$.

So, we have, for some constant k, $k\binom{n}{2}$ polynomials of degree at most 3 in $7n$ variables. Applying Warren's Theorem, we obtain that there are at most $n^{n(7+o(1))}$ graphs on n vertices which are intersection graphs of segments of planes. $\square$

2.5 Upper Bounds on Polynomials

We consider the upper bound of the number of intersection graphs of segments of polynomials of degree d in general. We use Sturm's Theorem [12]:

Given a univariate polynomial $p(x)$, define a *Sturm chain* as the following sequence of polynomials

$$\begin{aligned} p_0(x) &= p(x) \\ p_1(x) &= p'(x) \\ p_2(x) &= -\mathrm{rem}\left(p_0(x), p_1(x)\right) \\ &\ldots \\ p_s(x) &= -\mathrm{rem}\left(p_{s-2}(x), p_{s-1}(x)\right) \end{aligned}$$

where $\mathrm{rem}\left(p_i(x), p_j(x)\right)$ refers to the remainder of the polynomial long division of $p_i(x)$ by $p_j(x)$. Note that s is never greater than the degree of $p(x)$. Also note that this chain is definite for any general system of polynomials $a_d x^d + a_{d-1}x^{d-1} + \ldots + a_0$; given the degree of the polynomials, the remainders can be clearly defined in terms of the general variable coefficients $\{a_d, a_{d-1}, \ldots, a_0\}$. Let $\sigma(\xi)$ denote the number of positive or negative sign changes in the Sturm chain $p_0(\xi), p_1(\xi), \ldots, p_s(\xi)$.

Sturm's Theorem. *For any two real numbers a, b, the number of real roots of a polynomial p in the interval $(a, b\,]$ is given by $\sigma(a) - \sigma(b)$.*

The theorem is still applicable if the polynomial has multiple roots, on the condition that neither a nor b is a multiple root of p.

So, let $\mathscr{P} = \{P_1, P_2, \ldots, P_n\}$ be a system of segments of polynomials in the plane. Let each polynomial be of degree d. Let each polynomial $P_i(x)$ be constrained by $\ell_i \leq x \leq u_i$, for all $i = 1, 2, \ldots, n$.

Theorem 5. *The number of intersection graphs of n segments of polynomials of degree d is $n^{n(d+3+o(1))}$.*

Proof. Two polynomials P_i and P_j for $i, j \in \{1, 2, \ldots, n\}$ cross each other if and only if the roots of P_{ij} are within the constraints $\max\{\ell_i, \ell_j\} \leq x \leq \min\{u_i, u_j\}$. Applying Sturm's Theorem to P_{ij}, we obtain Sturm chains for each constraint, in which the number of sign changes in each determine whether or not there is a root and therefore whether the polynomials intersect. Multiplying every consecutive pair in each chain, the negative values represent the sign changes. Note that although each product is given solely in terms of the general variable coefficients of each polynomial, there may be functions of these coefficients in the denominators of the products. In order to use Warren's Theorem, we must make each of these denominators into a separate polynomial in the system (for the sign patterns), and then multiply the product through by the denominators. As such, the positive or negative signs of each product will remain intact through the system. For some constant k, we have $k\binom{n}{2}$ polynomials of degree at most $2d-1$ in $(d+3)n$ variables. By Warren's Theorem, we obtain that there are at most $n^{n(d+3+o(1))}$ graphs on n vertices which are intersection graphs of segments of polynomials of degree d. $\square$

2.6 Upper Bounds on Rational Functions

We extend the construction for the upper bounds on the number of intersection graphs of segments of polynomials to segments of rational functions.

Let $\mathscr{F} = \{F_1, F_2, \ldots, F_n\}$ be a system of segments of rational functions in the plane. Let each rational function be of the form

$$F_i : y = \frac{P_i(x)}{Q_i(x)} \text{ where } \ell_i \leq x \leq u_i \text{ for } i = 1, 2, \ldots, n$$

where $Q_i(x)$ is not the zero polynomial. Assume that $P_i(x)$ and $Q_i(x)$ have no common factors for all $i = 1, 2, \ldots, n$ and assume that $Q_i(x)$ and $Q_j(x)$ have no common factors for all $i \neq j$. Let d_P be the maximum degree of all $P_i(x)$ and d_Q be the maximum degree of all $Q_i(x)$.

Theorem 6. *The number of intersection graphs of n segments of rational functions, where the degree of the numerator is d_P and the degree of the denominator is d_Q, is $n^{n(d_P+d_Q+3+o(1))}$.*

Proof. Two rational functions F_i and F_j for $i,\ j \in \{1, 2, \ldots, n\}$ cross each other if and only if the roots of F_{ij} are within the constraints $\max\{\ell_i, \ell_j\} \leq x \leq \min\{u_i, u_j\}$. Note that $F_{ij} = \frac{P_iQ_j - P_jQ_i}{Q_iQ_j}$, so the roots of F_{ij} are given by the roots of $P_iQ_j - P_jQ_i$ such that $Q_iQ_j \neq 0$. If, for some value $x = k$, $Q_iQ_j = 0$, then without loss of generality $Q_i(k) = 0$. As such, $P_i(k)Q_j(k) - P_j(k)Q_i(k) = P_i(k)Q_j(k)$. Thus, in order for k to be a root of F_{ij}, since P_i and Q_i have no common factors, $Q_j(k) = 0$. This contradicts the fact that Q_i and Q_j have no common factors, so k cannot be a root of F_{ij}. So, two rational functions F_i and F_j for $i, j \in \{1, 2, \ldots, n\}$ cross each other if and only if the roots of $P_iQ_j - P_jQ_i$ are within the constraints $\max\{\ell_i, \ell_j\} \leq x \leq \min\{u_i, u_j\}$. Applying Sturm's Theorem to $P_iQ_j - P_jQ_i$, we obtain Sturm chains for each constraint, in which the number of sign changes in each determine whether or not the there is a root and as such if the rational functions intersect. Multiplying every consecutive pair in each chain, the negative values represent the sign changes. From here, we can use Warren's Theorem to receive an upper bound on the number of different signs and as such the possible intersection graphs. We have, for some constant k, $k\binom{n}{2}$ polynomials of degree at most $2d_P + 2d_Q - 1$ in $(d_P + d_Q + 3)n$ variables. We obtain that there are at most $n^{n(d_P+d_Q+3+o(1))}$ graphs on n vertices which are intersection graphs of segments of rational functions. $\square$

3 Lower Bounds on the Number of Intersection Graphs

We obtain lower bounds for various families of intersection graphs, starting with parabolas and using a similar construction for rotated parabolas. We extend these results more generally to polynomials and rational functions.

3.1 Definitions

First we introduce some key notation that will be used throughout the rest of this section. We begin by introducing *fences*, which are constructed in such a manner that we can choose specific sets of segments to intersect when adding more segments to a system. Note that we use line segments in these fences even when considering lower bounds on the number of intersection graphs of other functions, since the constructions in this section do not rely on the segments being straight.

Definition 6. A set of t line segments, each labeled s_k for some $k \in \{1,2,\ldots,t\}$, is called a *fence* if there are two vertical lines ℓ_1, ℓ_2 such that every line segment has one endpoint on ℓ_1 and one endpoint on ℓ_2, and for each set $\{i,i+1,\ldots,j\}$ we can find an x_{ij}, y_{ij} such that $\{i,i+1,\ldots,j\}$ is exactly the set of all k such that $s_k \cap \{(x,y)|x = x_{ij} \text{ and } y \leq y_{ij}\} \neq \emptyset$, as in Figure 1.

In this case, we label the line segments $\{1,2,\ldots,\varepsilon n\}$ for small ε.

Lemma 7. *There exist fences of any size.*

Proof. We prove this theorem by induction. For the base case, if we have 1 line segment, then we simply draw a line segment from $(0,0)$ to $(w,0)$ and we are done because any vertical line through the fence will cross that line segment.

Now, assume we have a fence with k line segments. For every j, we rename all line segments labeled j with $j+1$, so we want to add the line segment labeled 1. If necessary, we move the whole structure up so that one end of the line segment 1 can be located on the vertical axis starting at $(0,0)$ and below every other end point on the fence. Call x^* the least $x > 0$ such that there is an intersection of some pair of lines from $\{2,3,\ldots,k+1\}$. So, $x^* > 0$ and we can pick a large enough positive slope m for line segment 1 such that line segment 1 intersects every other segment before x^*. In the interval $[x^*,w]$, the fence has the same property as before for the lines $\{2,3,\ldots,k+1\}$. Moreover, every set of the form $\{1,2,\ldots,j\}$ can be found on the interval $[0,x^*]$. $\square$

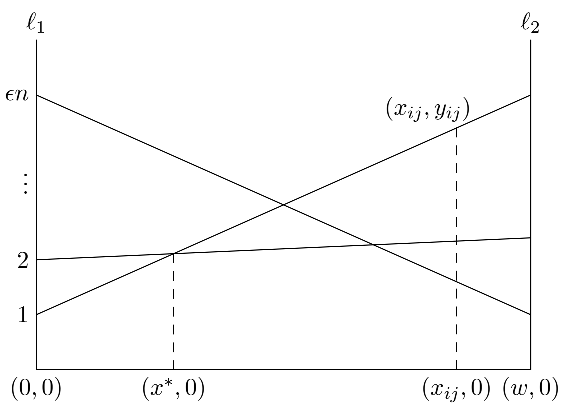

Figure 1: Fence example

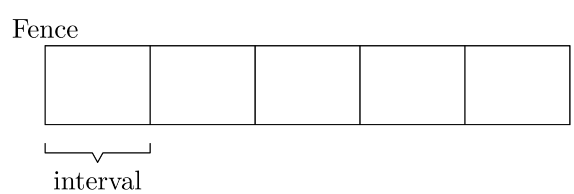

Figure 2: Fence and interval representation

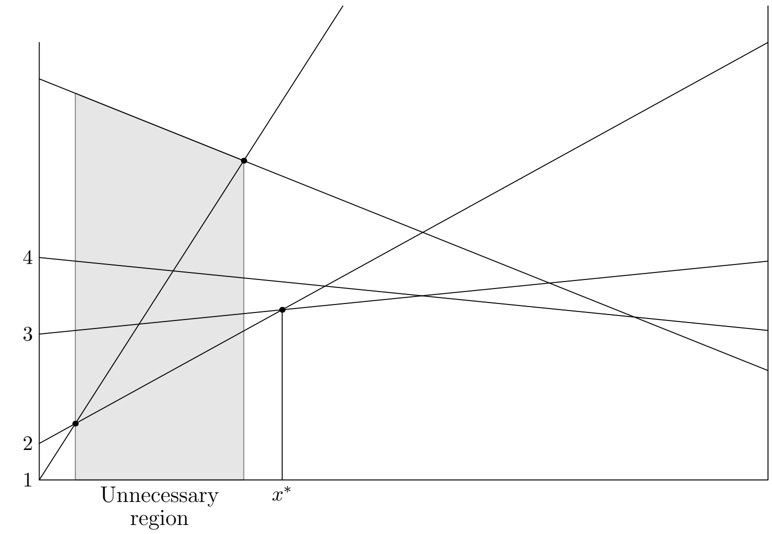

Figure 3: Unnecessary regions

We denote any range (x_1, x_2) where the vertical line segment from $(x, 0)$ to (x, y) for some large enough y and for all $x \in (x_1, x_2)$ intersects exactly the same line segments $\{i, i+1, \ldots, \varepsilon n\}$ as an *interval*. We represent a fence and the intervals as in Figure 2. Note that we can construct the fences so that the intervals are all of equal size. If we want a fence with m line segments and as such m intervals, we add each new line segment 1 as based on the inductive step of the proof above such that the last crossing (right-most) of the new line segment with any of the segments $\{2, 3, \ldots, k\}$ and x^* is $\frac{1}{m}$. There will be an *unnecessary region* between the first crossing (left-most) and last crossing of the line segment 1 and any of the segments $\{2, 3, \ldots, k\}$, in that without such a region the structure will still satisfy all of the requirements of a fence. We can make this region as small as necessary by simply moving the endpoint of line segment 1 on the y-axis down, until the distance between the first and last crossing is insignificant. As such, we can approximate a fence without these unnecessary regions, as in Figure 2. An unnecessary region is shaded in Figure 3.

Note that for each of the constructions in this section, we use fences of size εn for some small ε. In the constructions in this section, we obtain lower bounds of the form $(\varepsilon n)^{fn(1-2\varepsilon)}$. It is sufficient to let $\varepsilon = 1/\log n$. More optimally, $\varepsilon = \log\log n/\log n$.

3.2 Lower Bounds on Parabolas

In this subsection we obtain the lower bounds on the number of intersection graphs of segments of parabolas with vertical axes of symmetry.

Theorem 8. *The number of intersection graphs of n segments of parabolas with vertical axes of symmetry is $n^{n(5-o(1))}$.*

Proof. We start with two fences. Without loss of generality, we construct a parabola between the fences as in Figure 4. Note that the fences are facing each other, so the top fence is drawn as constructed in Section

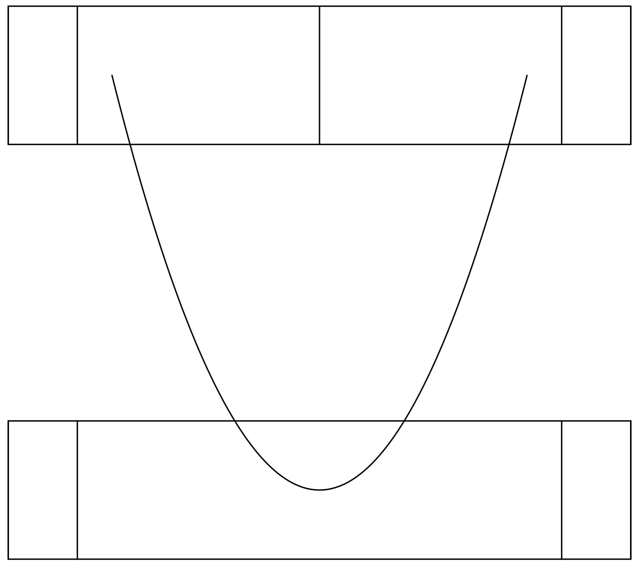

Figure 4: Parabola construction

3.1 while the bottom fence is constructed upside-down. We can use fences for the intersection graph of parabolas since as x goes to infinity or negative infinity, the equation of the parabola becomes practically linear. We can see the fences as constructed using these distinct sections of parabolas. Moreover, we are considering a lower bound, so inserting such conditions on the parabolas is acceptable, as we would merely be undercounting. We must first prove that given $\ell \to \infty$, each of the sections where the parabola intersects the fences can lie within one interval of the fence.

Consider the worst possible orientations of the parabola, as in Figures 5 and 6. For the first case, in Figure 5, we define "worst" in terms of the largest possible distance d_1 for any parabola drawn through these fences. The parabola is given by the points $(0, \ell - h)$, $(w, \ell - h)$, and $\left(\frac{w}{2}, 0\right)$. The equation of the parabola becomes

$$y = \frac{4(\ell - h)}{w^2}x^2 + \frac{4(h-\ell)}{w}x + \ell - h.$$

Taking $\ell \to \infty$, we obtain

$$\lim_{\ell \to \infty} d_1 = \lim_{\ell \to \infty} \frac{w\sqrt{-7h^2 + 10h\ell - 3\ell^2}}{2(\ell - h)} = 0$$

for d_1 as in Figure 5. For the second case, in Figure 6, we define "worst" in terms of the largest possible distances d_2 for any parabola drawn through these fences. The parabola is given by the points $(0, \ell)$ and $(w, \ell - h)$, with a vertex on $y = h$. The equation of the parabola becomes

$$y = \frac{-3h + 2\ell + 2\sqrt{(\ell - 2h)(\ell - h)}}{w^2}\left(x - \frac{w\left(-h + \ell - \sqrt{(\ell - 2h)(\ell - h)}\right)}{h}\right)^2 + h.$$

Therefore,

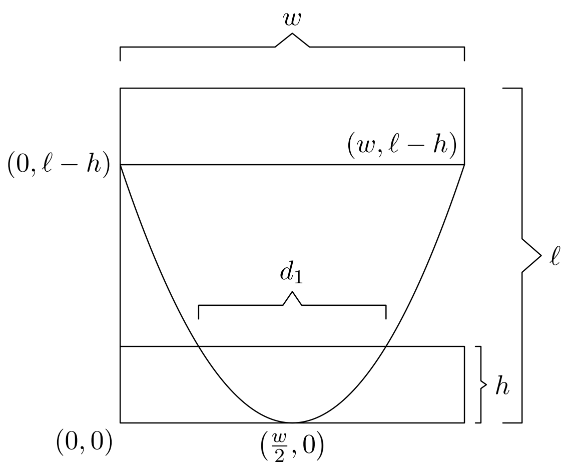

Figure 5: Worst case parabola construction for d_1

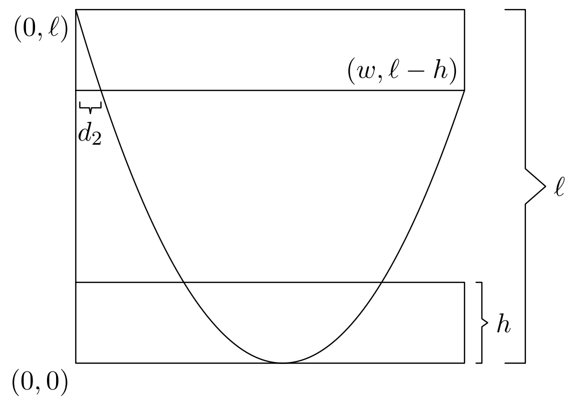

Figure 6: Worst case parabola construction for d_2

$$\lim_{\ell\to\infty} d_2 = \lim_{\ell\to\infty} \frac{hw}{-3h+2\ell+2\sqrt{(\ell-2h)(\ell-h)}} = 0$$

for d_2 as in Figure 6.

Thus, we can construct parabolas between these fences such that the intersections with the fences each lie within one interval. We must choose two ranges of the form $\{i, i+1, \ldots, j\} \in \{1, 2, \ldots, \varepsilon n\}$ and one depth of the form $k \in \{1, 2, \ldots, \varepsilon n\}$, which we can connect to form a parabola, as shown in the solid sections of Figure 7. The two ranges give the interval and depth in which the two ends of the parabola reach on the top fence. Since these two points already determine the interval in which the vertex of the parabola must lie on the bottom fence, we can only choose a depth on the bottom fence. Note that since the intervals line up between the two fences, once we choose one range and as such one interval on the top fence, we eliminate half of the intervals on the top fence so that the vertex of the parabola never intersects a boundary between the intervals in the bottom fence.

We have $\binom{\varepsilon n}{2}$ ways of choosing the first range, and $\binom{\frac{1}{2}\varepsilon n}{2}$ ways of choosing the second range. We have εn ways of choosing the depth, and we must multiply all of this by 2 to allow for the construction of parabolas facing the other way. We must construct $n(1-2\varepsilon)$ parabolas, giving a lower bound of $\left(2\binom{\varepsilon n}{2}\binom{\frac{1}{2}\varepsilon n}{2}(\varepsilon n)\right)^{n(1-2\varepsilon)} = n^{n(5-o(1))}$ for $\varepsilon = 1/\log n$. □

Note that the exponent here matches the exponent on the upper bounds for parabolas.

3.3 Lower Bounds on Rotated Parabolas

We consider the lower bounds on the number of intersection graphs of segments of rotated parabolas in the same manner as that of parabolas.

Theorem 9. *The number of intersection graphs of n segments of rotated parabolas is $n^{n(6-o(1))}$.*

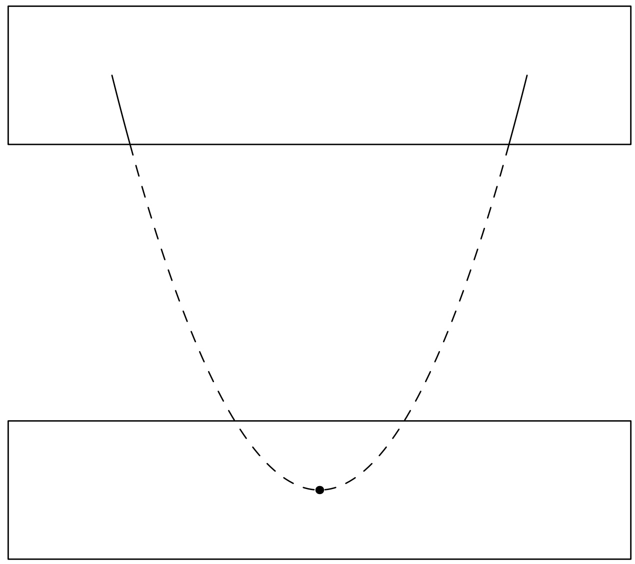

Figure 7: Choosing a parabola

Proof. We can assume that none of these parabolas have horizontal axes of symmetry. We can rotate the entire system and since there are only a finite number of segments, there will be a case where none of the axes of symmetry are horizontal. The intersection graph remains the same under rotation due to isomorphism. We start with two fences, and without loss of generality construct a parabola as in Figure 8. Note that the fences are facing each other, so the top fence is drawn as constructed in Section 3.1 while the bottom fence is constructed upside-down. We can use fences for the intersection graph of rotated parabolas since as x goes to infinity or negative infinity, the equation of the rotated parabola becomes practically linear. We can see the fences as constructed using these distinct sections of rotated parabolas. Moreover, we are considering a lower bound, so inserting such conditions on the rotated parabolas is acceptable, as we would merely be undercounting. We must first prove that each of the sections where the parabola intersects the fences can lie within one interval of the fence, given that $\ell \to \infty$.

Consider the worst possible orientation of the parabola, as in Figure 9. We define "worst" in terms of the largest possible distance d_1 for any rotated parabola drawn through these fences. The parabola is given by the points (r,ℓ), (s,ℓ), and $(0,0)$. Assume that $(0,0)$ is between r and s, so without loss of generality let r be negative and s be positive. Let $(0,0)$ be a local minimum. So, we can set the first derivative to 0 at $(0,0)$ to receive an equation in terms of just r, s, and ℓ. Note that in order to take the first derivative, we must consider the function in terms of y, which gives two cases by the quadratic formula. Only one of these cases is necessary, namely the positive case, since $(0,0)$ is not a point in the second case. As such, the equation of the parabola becomes

$$-\frac{\ell}{r+s}x^2+xy-\frac{r+s}{4\ell}y^2+\frac{(r-s)^2}{4(r+s)}y=0.$$

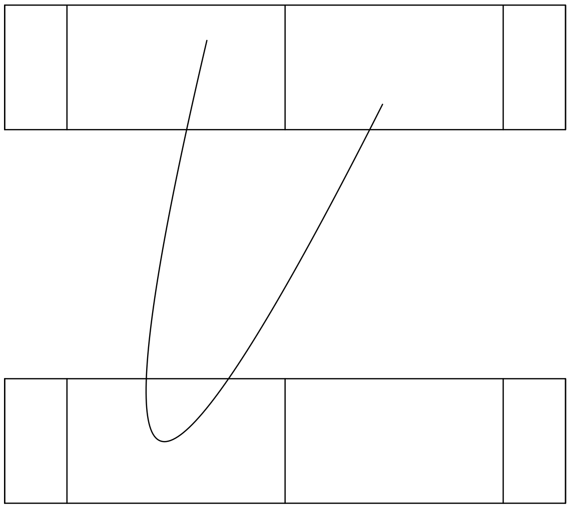

Figure 8: Rotated parabola construction

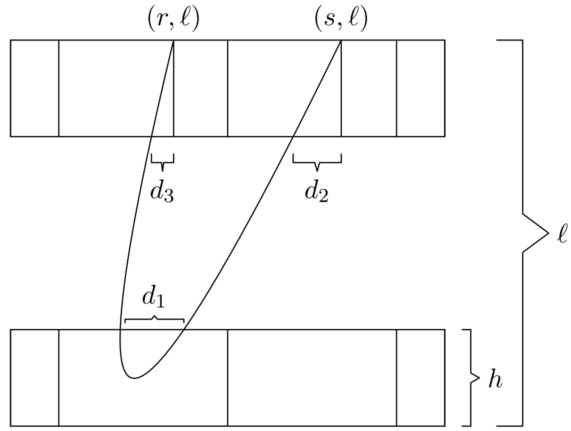

Figure 9: Worst case rotated parabola construction

As such,

$$\lim_{\ell\to\infty} d_1 = \lim_{\ell\to\infty} \frac{(r+s)\left(-h-\frac{r-s}{r+s}\sqrt{\ell h}\right)}{-2\ell} + \frac{(r-s)^2}{8\ell\,(r+s)} = 0 \text{ or}$$

$$\lim_{\ell\to\infty} d_1 = \lim_{\ell\to\infty} \frac{(r+s)\left(-h+\frac{r-s}{r+s}\sqrt{\ell h}\right)}{-2\ell} - \frac{y(r+s)+(r-s)\sqrt{\ell y}}{2\ell} = 0,$$

$$\lim_{\ell\to\infty} d_2 = \lim_{\ell\to\infty} s - \frac{(\ell-h)(r+s)-(r-s)\sqrt{\ell(\ell-h)}}{2\ell} = 0,$$

and

$$\lim_{\ell\to\infty} d_3 = \lim_{\ell\to\infty} r - \frac{(\ell-h)(r+s)+(r-s)\sqrt{\ell(\ell-h)}}{2\ell} = 0$$

for d_1,d_2, and d_3 as in Figure 9.

Note that in calculating d_1, we must calculate the minimum x, and must set the first derivative of the function in terms of x to 0. This again gives two cases by the quadratic formula, both of which are presented above.

Thus, we can construct parabolas between these fences such that the intersections with the fences each lie within one interval. We must choose three ranges of the form $\{i, i+1, \ldots, j\} \in \{1, 2, \ldots, \varepsilon n\}$, which we can connect to form a parabola, as shown in the solid sections of Figure 10. Note that this is different from the parabolas case in that the minimum does not have to lie exactly in the middle of r and s. As such, after we choose two ranges and thus the interval and depth in which the two ends of the parabola reach on the top fence, we still have the freedom to choose the interval and depth of the minimum.

We have $\binom{\varepsilon n}{2}$ ways of choosing each range. We must construct $n(1-2\varepsilon)$ parabolas, giving a lower bound of $\left(\binom{\varepsilon n}{2}^3\right)^{n(1-2\varepsilon)} = n^{n(6-o(1))}$ for $\varepsilon = 1/\log n$. $\square$

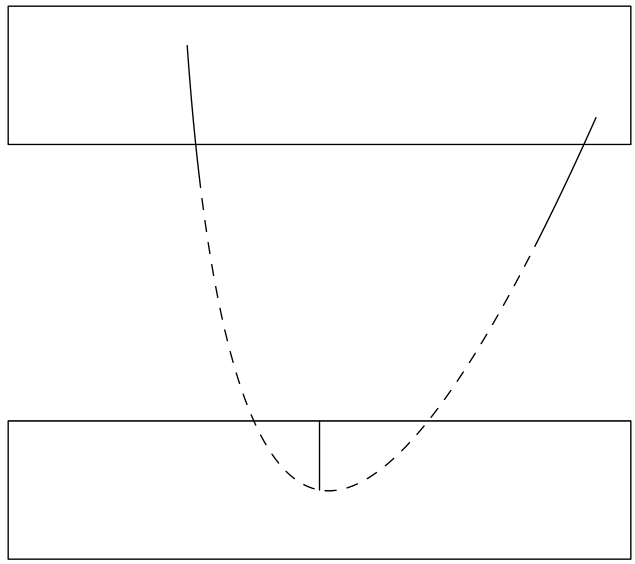

Figure 10: Choosing a rotated parabola

Note that the exponent here matches the exponent on the upper bounds on intersection graphs of rotated parabolas.

3.4 Lower Bounds on Polynomials

We consider the lower bounds on the number of intersection graphs of segments of polynomials of degree d in general.

Theorem 10. *The number of intersection graphs of n segments of polynomials of degree d is* $n^{n(d+3-o(1))}$.

Proof. We start with two fences. Without loss of generality, we construct a polynomial of degree d such that the ends of the segment lie on the fences, as in Figure 11. Note that the fences are facing each other, so the top fence is drawn as constructed in Section 3.1 while the bottom fence is constructed upside-down. We can use fences for the intersection graph of polynomials since as x goes to infinity or negative infinity, the equation of the polynomial becomes practically linear. We can see the fences as constructed using these distinct sections of polynomials. Moreover, we are considering a lower bound, so inserting such conditions on the polynomials is acceptable, as we would merely be undercounting. We must first prove that each of the sections where the polynomial intersects the fences can lie within one interval of the fence, given that $l \to \infty$.

Let $\mathscr{P} = \{P_1, P_2, \dots, P_n\}$ be a system of segments of polynomials in the plane. Let each polynomial be of degree d. Consider some λ such that $\lambda > 1$. The system of segments

$$N_i : y = \lambda P_i \text{ for all } i = 1, 2, \dots, n$$

gives an equivalent intersection graph to the original system $\mathscr{P}$. So, considering the first derivative of $x + \varepsilon$

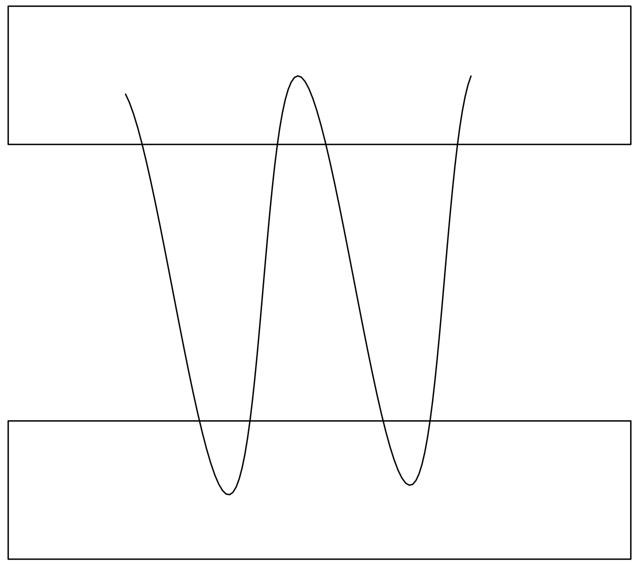

Figure 11: Polynomial construction

for small ε,

$$N_i'(x+\varepsilon) = \lambda P_i'(x+\varepsilon) \geq \lambda \nu > 0$$

where ν is a constant independent of the value of x. As such, each polynomial N_i can have a slope that is essentially vertical, allowing the ends of the segments to lie within one interval of the fence.

Thus, we can construct polynomials between these fences such that the intersections with the fences each lie within one interval. We must choose two ranges of the form $\{i, i+1, \ldots, j\} \in \{1, 2, \ldots, \varepsilon n\}$ and $d-1$ depths of the form $k \in \{1, 2, \ldots, \varepsilon n\}$, which we can connect to form a polynomial, as shown in the solid sections of Figure 12. The two ranges give the interval and depth in which the two ends of the polynomial reach on either the top or bottom fence. Since these two points already determine the interval in which the local maxima and minima of the polynomial must lie, we can only choose depths for those points.

So, we have $\binom{\varepsilon n}{2}$ ways of choosing each range and εn ways of choosing each depth. We must construct $n(1-2\varepsilon)$ polynomials, giving us a lower bound of $n^{n(d+3-o(1))}$ for $\varepsilon = 1/\log n$. □

Note that the exponent here matches the exponent on the upper bounds on intersection graphs of polynomials.

3.5 Lower Bounds on Rational Functions

We extend the construction for the lower bounds on the number of intersection graphs of segments of polynomials to segments of rational functions.

Let $\mathscr{F} = \{F_1, F_2, \ldots, F_n\}$ be a system of segments of rational functions in the plane. Let each rational function be of the form

$$F_i : y = \frac{P_i(x)}{Q_i(x)} \text{ where } \ell_i \leq x \leq u_i \text{ for } i = 1, 2, \ldots, n$$

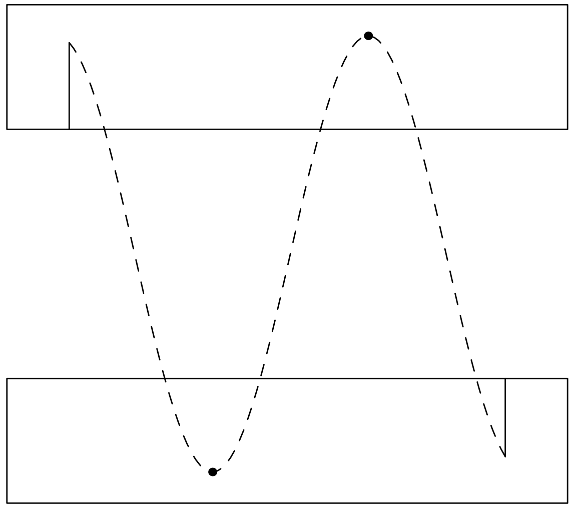

Figure 12: Choosing a polynomial

where $Q_i(x)$ is not the zero polynomial. Let d_P be the maximum degree of all $P_i(x)$ and d_Q be the maximum degree of all $Q_i(x)$.

Theorem 11. *The number of intersection graphs of n segments of rational functions, where the degree of the numerator is d_P and the degree of the denominator is d_Q, is $n^{n(d_P+d_Q+3-o(1))}$.*

Proof. We start with two fences, in the same manner as in the polynomial case above. Note that the fences are facing each other, so the top fence is drawn as constructed in Section 3.1 while the bottom fence is constructed upside-down. We can use fences for the intersection graph of rational functions since as x goes to infinity or negative infinity, the equation of the rational function becomes practically linear. We can see the fences as constructed using these distinct sections of rational functions. Moreover, we are considering a lower bound, so inserting such conditions on the functions is acceptable, as we would merely be undercounting. Also note that for the same reason as in the polynomial case above, each rational function can have a slope that is essentially vertical as it intersects the fences, allowing the ends of the segments lie within one interval of the fence.

Thus, we can construct rational functions between these fences such that the intersections with the fences each lie within one interval. We must choose the depths to which the local extrema of each rational function intersects the fences. We receive the local extrema by taking the first derivative of the rational function; as such, the roots of $P_i'(x)Q_i(x) - P_i(x)Q_i'(x)$ give the local extrema. Assume that the roots are not in the denominator of the first derivative, namely roots of $Q_i(x)^2$. Also assume that every root is real and gives a local extrema. As such, there are $d_P + d_Q - 1$ extrema and we must choose $(\varepsilon n)^{d_P+d_Q-1}$ depths for each rational function. We must also choose two ranges for each function, which give the interval and depth in which the two ends of the rational function reach on either the top of the bottom fence. Note that in doing this, we assume that $d_P > d_Q$, so we ignore horizontal asymptotes. Moreover, note that the intervals in which the vertical asymptotes lie are accounted for due to the local extrema, so it is unnecessary to count the vertical asymptotes, and note that we are assuming there are no point discontinuities. So, we have $\binom{\varepsilon n}{2}$ ways of choosing each range. We must construct $n(1-2\varepsilon)$ rational functions, giving us a lower bound of $n^{n(d_P+d_Q+3-o(1))}$ for $\varepsilon = 1/\log n$. $\square$

Note that the exponent here matches the exponent on the upper bounds on intersection graphs of rational functions.

4 Conclusion

We have obtained nearly tight upper and lower bounds on the speeds of graphs of n vertices which are intersection graphs of parabolas, rotated parabolas, conic sections, planes in $\mathbb{R}^3$, polynomials, and rational functions, using a variety of techniques involving Warren's Theorem for the upper bounds and constructive methods involving fences for the lower bounds. In general, the bounds on the speeds of intersection graphs of these systems is $n^{n(f+o(1))}$, where f is the degree of freedom.

In the future, we hope to obtain tight bounds on the speeds of intersection graphs of the more general case of algebraic curves and semi-algebraic sets, using similar techniques. We conjecture that the number of intersection graphs of these systems are of the same form $n^{n(f+o(1))}$, where f is the degree of freedom.

5 Acknowledgments

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