1)

Modular representations. (after Carter-Lusztig).

In the definition of a representation $\rho : G \to GL(E)$ we now replace $E/\mathbb{C}$ by a vector space E over a fixed algebraically closed field K of characteristic $p > 0$. One obtains the notion of modular representation. It is no longer true that a G-stable subspace of E has a G-stable complement. The character of E as a function $G \to K$ can be defined but it is not very useful, it does not determine the isomorphism class of the representation.

We want to study modular representations of $G = GL(V)$ where V is an n-dim. vector space over F_q, a finite field with q elements, q a power of p. As before $\mathcal{B}$ is the set of complete flags $V_* = (V_0 \subset V_1 \subset \dots \subset V_n)$ in V with transitive action of G. Let $\mathcal{F}$ be the K-vector space of functions $\mathcal{B} \to K$. It is a modular rep. of G: If $g \in G$, $f \in \mathcal{F}$ then $(gf)(V_0 \subset \dots \subset V_n) = f(g^{-1}(V_0), \dots, g^{-1}(V_n))$.

2)

As before $\mathcal{B} \times \mathcal{B}$ is a disjoint union $\bigcup \mathcal{O}_\sigma$ of G-orbits (σ runs through S the symmetric group). As before, the Hecke algebra $\mathcal{H}_K$ is the vector space of all functions $T : \mathcal{B} \times \mathcal{B} \to K$ which are constant on G-orbits. It has a K-basis $\{T_\sigma \mid \sigma \in S\}$ where $T_\sigma : \mathcal{B} \times \mathcal{B} \to K$ is 1 on $\mathcal{O}_\sigma$ and 0 on $\mathcal{O}_{\sigma'}$, $\sigma' \neq \sigma$. Now $\mathcal{H}_K$ has an algebra structure $T * T'$ defined just as for $\mathcal{H}/\mathbb{C}$: We have again $T_\sigma * T_{\sigma'} = T_{\sigma\sigma'}$ if $|\sigma\sigma'| = |\sigma| + |\sigma'|$,
$T_{\sigma_i}(T_{\sigma_i} + 1) = 0$, for $i \in \{1, .., n-1\}$.
(As in $\mathcal{H}$, we have $(T_{\sigma_i} - q)(T_{\sigma_i} + 1) = 0$, but $q = 0$ as an element of K.)

We define an algebra homomorphism
$\mathcal{H}_K \to \mathrm{End}_G(\mathcal{F})$ by $T : [f \to Tf]$
where $f \in \mathcal{F}, Tf = \mathcal{F}$ are related by

$$(Tf)(V_*) = \sum_{V'_* \in \mathcal{B}} T(V_*, V'_*)\, f(V'_*) \, .$$

Here $\mathrm{End}_G(\mathcal{F})$ is the algebra of linear maps $\mathcal{F} \to \mathcal{F}$ which commute with the G-action on $\mathcal{F}$.

3)

Let w_0 be the element of S given by $1 \to n$, $2 \to n-1, \ldots, n \to 1$. For any $i<j$ in $\{1, \ldots, n\}$ we have $w_0(i) > w_0(j)$. Hence $|w_0| = \binom{n}{2}$. We show:

If $\sigma \in S$ then $|\sigma| + |\sigma^{-1} w_0| = |w_0|$ (*)

Recall:

$$I_\sigma = \{(i<j) \mid \sigma(i) > \sigma(j)\}$$

$$I_{\sigma^{-1} w_0} = \{(i<j) \mid \sigma^{-1}(n+1-i) > \sigma^{-1}(n+1-j)\}$$

$$= \{i' > j' \mid \underbrace{\sigma^{-1}(i')}_{i''} > \underbrace{\sigma^{-1}(j')}_{j''}\}$$

$$= \{j'' < i'' \mid \sigma(j'') < \sigma(i'')\}.$$

Thus

$$|\sigma| + |\sigma^{-1} w_0| = \# I_\sigma + \# I_{\sigma^{-1} w_0} = \#(i<j) = |w_0|.$$

Let $J \subset \{1, 2, \ldots, n-1\}$. Recall that S_J is the subgroup of S generated by $\{s_i \mid i \in J\}$.

There is a unique element $\varpi_J \in S_J$ such that $|\sigma| < |\varpi_J|$ for any $\sigma \in S_J$, $\sigma \neq \varpi_J$, we have $\varpi_J^2 = 1$.

For any $\sigma \in W_J$ we have

$$|\sigma \varpi_J w_0| = |\sigma| + |\varpi_J w_0|.$$

or equivalently $|w_0| - |w_J \sigma^{-1}| = |\sigma| + |w_0| - |\varpi_J|$

or equivalently $|\varpi_J| = |\varpi_J \sigma^{-1}| + |\sigma|.$

4)

This follows from (*) applied to a product of symmetric groups instead of S.

We define

$$\Theta_J = \sum_{\sigma\in S_J} T_\sigma T_{\sigma_J w_\emptyset} = \sum_{\sigma\in S_J} T_{\sigma\sigma_J w_\emptyset} \in \mathcal{H}_K .$$

We show:

$$(*) \quad T_{\sigma_i}\Theta_J = \begin{cases} 0 & \text{if } i\in J \\ -\Theta_J & \text{if } i\notin J. \end{cases}$$

Assume that $i\in J$. Then S_J can be partitioned into pairs $\{\sigma, \sigma_i\sigma\}$, $|\sigma| = |\sigma_i\sigma| - 1$. For each such pair we have $T_{\sigma_i}(T_\sigma + T_{\sigma_i\sigma}) =$
$= T_{\sigma_i}(T_{\sigma_i}T_\sigma + T_\sigma) = T_{\sigma_i}(T_{\sigma_i}+1)T_\sigma = 0$. This proves (*) when $i\in J$. Now assume that $i\notin J$.

It is enough to show for $\sigma\in S_J$:

$$T_{\sigma_i} T_{\sigma\sigma_J w_\emptyset} = -T_{\sigma\sigma_J w_\emptyset}.$$

This would hold provided that $|\sigma_i\sigma\sigma_J w_\emptyset| < |\sigma\sigma_J w_\emptyset|$ or equiv. that $|\sigma_i\sigma\sigma_J| > |\sigma\sigma_J|$. But for any $\sigma'\in S_J$, $i\notin J$ we have $|\sigma_i\sigma'| > |\sigma'|$. This proves (*).

Conversely, we prove:

Let $\xi = \sum_{\tau \in S} c_\tau T_\tau \in \mathcal{H}_K$, $c_\tau \in K$ be such that

$$T_{\sigma_i}\xi = \begin{cases} 0 & \text{if } i \in J \\ -\xi & \text{if } i \notin J \end{cases}.$$

Then ξ is a scalar times Θ_J.

Assume that $i \in J$. We partition S into pairs $\{\tau, \sigma_i\tau\}$ with $|\tau| < |\sigma_i\tau|$. We have

$$T_{\sigma_i}(c_\tau T_\tau + c_{\sigma_i\tau} T_{\sigma_i\tau}) = c_\tau T_{\sigma_i\tau} - c_{\sigma_i\tau} T_{\sigma_i\tau} = 0$$

Thus $c_\tau = c_{\sigma_i\tau}$ for any $i \in I$. It follows that c_τ is constant for τ in any coset $S_J \backslash S$. Assume now that $c_\tau \neq 0$ for some $i \notin J$ such that $|\sigma_i\tau| > |\tau|$. We have

$$T_{\sigma_i}(c_\tau T_\tau + c_{\sigma_i\tau} T_{\sigma_i\tau}) = c_\tau T_{\sigma_i\tau} - c_{\sigma_i\tau} T_{\sigma_i\tau}$$

$$= -c_\tau T_\tau - c_{\sigma_i\tau} T_{\sigma_i\tau}$$

so that $c_\tau = 0$, contradiction. Thus $c_\tau \neq 0$ implies $|\sigma_i\tau| < |\tau|$ for any $i \notin J$, so that $|\sigma_i \tau w_0| > |\tau w_0|$ for any $i \notin J$. By the first part

6) we can assume that τ has maximal length in its S_J coset so that $|\sigma_{i'}\tau| < |\tau|$ for any $i' \in J$, hence $|\sigma_{i'}\tau w_0| > |\tau w_0|$ for any $i' \in J$. But then $|\sigma_j \tau w_0| > |\tau w_0|$ for any j and this implies $\tau w_0 = 1$ and $\tau = w_0$. We see that $\xi = \sum_{\sigma \in S_J} T_\sigma w_0 = \Theta_J$. $\square$

Let $\mathcal{F}_J = \text{image}(\Theta_J : \mathcal{F} \to \mathcal{F})$.

Since Θ_J commutes with the action of G on $\mathcal{F}$, we see that $\mathcal{F}_J$ is a G-stable subspace of $\mathcal{F}$.

We fix $B_0 \in \mathcal{B}$.

(elem. with all eigenvalues 1.) Let $\varphi \in \mathcal{F}$ be the function $\mathcal{B} \to K$ whose value at B_0 is 1 and whose value at any point $\neq B_0$ is 1. Since the G action on $\mathcal{B}$ is transitive, the elements $\{g\varphi \mid g \in G\}$ span $\mathcal{F}$. Hence the elements $\{g\Theta_J(\varphi) \mid g \in G\}$ span $\mathcal{F}_J$. The B_0-orbits on $\mathcal{B}$ are $\mathcal{O}'_\sigma = \{B \in \mathcal{B} \mid (B, B_0) \in \mathcal{O}_\sigma\}, \sigma \in S$.

Now ${}^\sigma\varphi : \mathcal{B} \to K$, $(\sigma \in S)$ ${}^\sigma\varphi = \begin{cases} 1 & \text{on } \mathcal{O}'_\sigma \\ 0 & \text{on } \mathcal{B} - \mathcal{O}'_\sigma \end{cases}$ are B_0-invariant elements of $\mathcal{F}$; they form a basis for the vector space $\{f \in \mathcal{F} \mid b_0 f = f, \ \forall\, b_0 \in B_0\} = \mathcal{F}^{B_0}$.

We have $T_\sigma \varphi = {}^\sigma\varphi$ for any $\sigma \in S$. Hence $\{T_\sigma \varphi \mid \sigma \in S\}$ is a basis of $\mathcal{F}^{U_0}$. We show:

(*) The vector space of B_0-invariant elements in $\mathcal{F}_J$ is the one dimensional space spanned by $\oplus_J \varphi$.

For any $f \in \mathcal{F}_J$ we have $T_i f = \begin{cases} 0 & \text{if } i \in J \\ -f & \text{if } i \notin J. \end{cases}$ (see above).

In particular this holds for $f = \oplus_J \varphi$. Since φ is U_0-invariant and $\oplus_J$ commutes with the G-action, we see that $\oplus_J \varphi$ is B_0-invariant.

Now any B_0-invariant element in $\mathcal{F}_J$ is of the form $\xi = \sum_\sigma c_\sigma T_\sigma \varphi$ where $c_\sigma \in K$ and $T_i \xi = \begin{cases} 0 & \text{if } i \in J \\ -\xi & \text{if } i \notin J \end{cases}$ hence $T_i\left(\sum c_\sigma T_\sigma\right) = \begin{cases} 0 & \text{if } i \in J \\ -\sum c_\sigma T_\sigma & \text{if } i \notin J. \end{cases}$

Hence $\sum c_\sigma T_\sigma = c \oplus_J$ for some $c \in K$ (see above).

This proves (*).

Let U_0 be the set of elements $g \in B_0$ such that g has all eigenvalues $=1$. It is a subgroup of order power of q.

8)

Let $M \subset \mathcal{F}_J$ be a non-zero G-stable subspace. Then there must exist a non-zero U_0-invariant vector $\xi \in M$. See Lemma below. Also ξ is necessarily B_0-invariant (see Lemma) $\Rightarrow \xi = c\,\Theta_J \varphi$ for some $c \in K$. Since $\{g\,\Theta_J \varphi \mid g \in G\}$ span $\mathcal{F}_J$ and are all in M, we see that $\mathcal{F}_J \subset M$ hence $\mathcal{F}_J = M$. Thus $\mathcal{F}_J$ is irreducible.

Lemma. Let $\rho : U \to GL(M)$ be a lin representation of a finite group U of order power of p, where M is a K-vector space of finite dim. Then there exists $x \in M - 0$ such that $\rho(u)x = x$ for any $u \in U$.

Proof. Let M_0 be the F_p-vector subspace of M generated by $\{\rho(u)x \mid u \in U\}$. It has $p^n > 1$ elements. It is a union of U-orbits. Each orbit has cardinal 1 or a multiple of p. Let N be the number of orbits with a single element. Then $N +$ multiple of $p = p^n$. Hence N is divisible by p. Hence $N \geq 2$. Hence we can choose $x \neq 0$. $\square$

Lemma Any U_0-invariant function in $\mathcal{F}$ is automatically B_0-invariant.

Proof. Enough to prove: the obvious surjective map $U_0 \backslash G/B_0 \longrightarrow B_0 \backslash G/B_0$ is a bijection, that is, if g, g' in G satisfy $g' \in B_0 g B_0$, then $g' \in U_0 g B_0$. We have $g \in B_0 \sigma B_0$, $g' \in B_0 \sigma B_0$ for some $\sigma \in S$. (We identify B_0 with upper triangular matrices in G, S with permutation matrices in G, U_0 with upper triang. matrices with 1 on diagonal.) We have $B_0 = U_0 \cdot T$, where T are the diagonal matrices, so that $T\sigma = \sigma T$. Thus $g \in U_0 \sigma T B_0 = U_0 \sigma B_0$, $g' \in U_0 \sigma T B_0 = U_0 \sigma B_0$ and $g' \in U_0 g B_0$. $\square$

Proposition. Any irreducible G-stable subspace of $\mathcal{F}$ is isomorphic to $\mathcal{F}_J$ (for some J) as a representation of G.

[one can show that the words "is isomorphic to" in the Prop. can be replaced by "is equal to"; the words "for some J" can be replaced by "for a unique J".]

Proof. Let M be an irreducible G-stable subspace of $\mathcal{F}$. The set $\{\sigma \in S;\, T_\sigma M \neq 0\}$ is non-empty. (It contains for example 1.) Hence we can select σ in this set with $|\sigma|$ maximum possible. We show that if $i \in \{1, \dots, n-1\}$ then $T_\sigma M$ is contained in $\{f \in \mathcal{F} \mid T_{\sigma_i} f = 0\}$ or in $\{f \in \mathcal{F} \mid T_{\sigma_i} f = -f\}$. Assume first that $|\sigma_i \sigma| > |\sigma|$. Then $T_{\sigma_i} T_\sigma M = T_{\sigma_i \sigma} M = 0$ by the maximality of $|\sigma|$; thus $T_\sigma M \subset \{f \in \mathcal{F} \mid T_{\sigma_i} f = 0\}$. Assume next that $|\sigma_i \sigma| < |\sigma|$. Then $\sigma = \sigma_i \sigma'$, $|\sigma| = |\sigma'| + 1$ and for $m \in M$ we have $T_{\sigma_i} T_\sigma m = T_{\sigma_i} T_{\sigma_i} T_{\sigma'} m =$ $= - T_{\sigma_i} T_{\sigma'} m = - T_\sigma m$; thus $T_\sigma M \subset \{f \in \mathcal{F} \mid T_{\sigma_i} f = -f\}$, as claimed. Since M is irreducible, the map $M \to T_\sigma M$ given by T_σ is an isomorphism of representations. Thus we can assume that M is contained in the kernel of T_{σ_i} or in the kernel of $T_{\sigma_i} + 1$ (for any i). As before, M contains some nonzero U_0-invariant (hence B_0-invariant vector) which by an earlier result must be a multiple of some $\Theta_J \varphi$. Hence $M = \mathcal{F}_J$. $\square$