1)

The Steinberg representation.

Let $G = GL(V)$ where V is an n-dim vector space $/F_q$. Let $\mathcal{B}$ be the set of complete flags in V. Then G acts transitively on $\mathcal{B}$ and the orbits of G on $\mathcal{B}\times\mathcal{B}$ are $\mathcal{O}_\sigma$, $\sigma \in S$, the group of permutations of $\{1,2,..,n\}$. Let $\mathcal{H}$ be the Hecke algebra (algebra of functions on $\mathcal{B}\times\mathcal{B}$ constant on G-orbits). It has basis $\{T_\sigma \mid \sigma\in S\}$. Let $\mathcal{F}$ be the vector space of functions $\mathcal{B}\to\mathbb{C}$. We have an algebra homomorphism (in fact isomorphism)

$\mathcal{H}\to \mathrm{End}_G(\mathcal{F})$, $T\to[f\to Tf]$,

$$(Tf)(B) = \sum_{B'\in\mathcal{B}} T(B,B')f(B') \text{ for } T\in\mathcal{H},\ f\in\mathcal{F}.$$

$\mathrm{End}_G(\mathcal{F})$ is the algebra of linear maps $\mathcal{F}\to\mathcal{F}$ commuting with the G-action on $\mathcal{F}$:

$g: f\to f'$, $f'(B) = f(g^{-1}(B))$.

Let $\Sigma = \{f\in\mathcal{F} \mid (T_{\sigma_i}+1)(f)=0 \text{ for } i=1,..,n\}$.
This is clearly a G-stable subspace of $\mathcal{F}$, called the Steinberg repres. of G.

2) We write $(B, B'):\sigma$ instead of $(B, B') \in \mathcal{O}_\sigma$

We fix $B_0 \in \mathcal{B}$. Let $\mathcal{B}_* = \{B \in \mathcal{B}; (B, B_0) : w_0\}$

Here $w_0 \in S$ is defined by $1 \to n$, $2 \to n-1, \dots,$ $n \to 1$. Let Σ_* be the vector space of functions $\mathcal{B}_* \to \mathbb{C}$. We show LEMMA.

The restriction $f \to f|_{\mathcal{B}*}$ is an isomorphism $\Sigma \xrightarrow{\sim} \Sigma_*$.

For any $f_* \in \Sigma_*$ we define $f \in \mathcal{F}$ by

$$f(B) = \varepsilon_B \sum_{B';\, (B'B):w_0\sigma^{-1}} f_*(B')$$

where $(B, B_0) : \sigma$, $\varepsilon_B = (-1)^{|\sigma w_0|}$. Note that each B' in the sum is automatically in $\mathcal{B}_*$.

For $i \in \{1, \dots, n-1\}$, we show

$(T_{\sigma_i} \cdot f)(B_1) + f(B_1) = 0$ for any B_1 with $(B_1, B_0) : \tau$

Equivalently:

3)

$$\sum_{\substack{\tilde B,\, B' \\ (B_1,\tilde B):\ \sigma_i \\ (B',\tilde B):\ w_0\tau^{-1}}} \varepsilon_{\tilde B}\, f_*(B') \;+\; \sum_{\substack{B' \\ (B',B_1):\ w_0\tau^{-1}}} \varepsilon_{B_1} f_*(B') \overset{?}{=} 0$$

where

$(\tilde B, B_0): \tau$

$$+ \sum_{\substack{\tilde B,\, B' \\ (B_1,\tilde B):\ \sigma_i \\ (B',\tilde B):\ w_0\tau^{-1}\sigma_i \\ (\tilde B,B_0):\ \sigma_i\tau}} \varepsilon_{\tilde B}\, f_*(B')$$

we must show that if

$(B', B_1): \ w_0\tau^{-1}$ (1)

or $w_0\tau^{-1}\sigma_i$ (2)

we have

$$\sum^{(\alpha)}_{\tilde B} \varepsilon_{\tilde B} + \sum^{(\beta)}_{\tilde B} \varepsilon_{\tilde B} + \left\{\begin{matrix} \varepsilon_{B_1} & \text{in case 1} \\ 0 & \text{in case 2} \end{matrix}\right\} \overset{?}{=} 0$$

$(B_1,\tilde B): \sigma_i \longleftrightarrow (B_1, \tilde B): \sigma_i$

$B' \underset{w_0\tau^{-1}}{\smile} \tilde B \underset{\tau}{\smile} B_0 \qquad B' \underset{w_0\tau^{-1}\sigma_i}{\smile} \tilde B \underset{\sigma_i\tau}{\smile} B_0$

We denote by C the unique $\tilde B$ such that $B' \underset{w_0\tau^{-1}}{\smile} C \underset{\tau}{\smile} B_0$

and by C' the unique $\tilde B$ such that $B' \underset{w_0\tau^{-1}\sigma_i}{\smile} C' \smile B_0$

4)

We must show

$$\varepsilon_C \begin{Bmatrix} 1 \text{ if } (C, B_1):\sigma_i \\ 0 \text{ if } \cancel{(C, B_1)}:\sigma_i \end{Bmatrix} + \varepsilon_{C'} \begin{Bmatrix} 1 \text{ if } (C', B_1):\sigma_i \\ 0 \text{ if } \cancel{(C', B_1)}:\sigma_i \end{Bmatrix}$$

$$+ \varepsilon_{B_1} \begin{Bmatrix} 1 & \text{in case } 1 \\ 0 & \text{in case } 2 \end{Bmatrix} = 0.$$

In case 1 we have $C = B_1$ and C' is such that $(C', B_1):\sigma_i$. The required equality is

$$0\,\varepsilon_{B_1} + 1\,\varepsilon_{C'} + \varepsilon_{B_1} = 0 \quad ; \text{ it is clear.}$$

Assume we are in case 2. We have $C \neq B_1$ since $B', B_1 : w_0 \tau^{-1} \sigma_i$, $(B', C) : w_0 \tau^{-1}$. We have $C' \neq B_1$ since $B_1, B_0 : \tau$, $C', B_0 : \sigma_i \tau$. Moreover we have $(C, C') : \sigma_i$. If $(B_1, C) : \sigma_i$ then using $(C, C') : \sigma_i$, $B_1 \neq C'$, we deduce $(B_1, C') : \sigma_i$. Similarly if $(B_1, C') : \sigma_i$ then $(B_1, C) : \sigma_i$. Thus

$(B_1, C) : \sigma_i \Leftrightarrow (B_1, C') : \sigma_i$

If both $(B_1, C) : \sigma_i$, $(B_1, C') : \sigma_i$, the required identity is $\varepsilon_C + \varepsilon_{C'} = 0$ which is clear. If both $(B_1, C) = \sigma_i$, $(B_1, C') = \sigma_i$ are false, the required identity is $0 + 0 = 0$. $\square$

5)

From the Lemma we have linear maps $R: \Sigma \to \Sigma_*$, $f \to f|_{\Sigma*}$, $R': \Sigma_* \to \Sigma$, $f_* \to f$ as in Lemma. Clearly, $RR' = 1$. Hence R is surjective. We show that R is injective. Let $f \in R$ be such that $f|_{\Sigma*} = 0$. For $B \in \mathcal{B}$ such that $(B\ B_0) : \sigma$ we show that $f(B)$ by descending induction on $|\sigma|$. When $|\sigma|$ is maximal, this holds by our assumption. Now assume that $\sigma \neq w_0$. We can find $i \in \{1, \dots, n-1\}$ such that $(B', B) : \sigma_i$ implies $(B', B_0) : \sigma_i \sigma$, $|\sigma_i \sigma| = |\sigma| + 1$. The sum of values of f over all such B' is equal to $-f(B)$. But $f(B') = 0$ by the ind. hypothesis, so that $f(B) = 0$. This proves the injectivity of R. We see that R is an isomorphism with inverse R'.

Lemma. The representation of G on Σ is irreducible

Proof We have $\Sigma \neq 0$ since $\Sigma_* \neq 0$. Assume that $\Sigma = \Sigma' \oplus \Sigma''$ where Σ', Σ'' are nonzero G-stable subspaces. We can find $\lambda \in End_G(\mathcal{F})$ such that $\lambda = 1$ on Σ', $\lambda = 0$ on Σ''. Since $End_G(\mathcal{F}) = \mathcal{H}$

we see that some element $\xi \in \mathcal{H}$ acts as 1 on E' and as 0 on E''. By definition $T_i \in \mathcal{H}$ acts on E as -1 for all i. It follows that any element of $\mathcal{H}$ (and in particular ξ acts on Σ as a scalar. This is a contradiction. The lemma is proved.