1)

Semi-direct products.

Let A be a finite abelian group, operation $+$
Let H be a finite group w— a given homomorphism

$$H \to \left\{ \begin{array}{l} \text{isomorphisms } A \xrightarrow{\alpha} A \\ \text{of groups} \end{array} \right\}$$

$$h \to [a \to h(a)].$$

Let $G = H \times A$ with product defined by

$$(h, a)(h', a') = (hh', h'^{-1}(a) + a').$$

Associativity

$$(h, a)(h', a')(h'', a'') =$$

$$(hh', h'^{-1}(a) + a')(h'', a'')$$

$$= (hh'h'', h''^{-1}(h'^{-1}(a)) + h''^{-1}(a') + a'')$$

$$\stackrel{?}{=} (h, a)(h'h'', h''^{-1}(a') + a'')$$

$$= (hh'h'', (h'h'')^{-1}(a) + h''^{-1}(a') + a'')$$

G is called the <u>semidirect</u> product of H, A with A normal: we have

$(h, a)(1, a') = (1, a'')(h, a)$ with

$(h, a+a') = (h, h^{-1}(a'') + a)$, so that $a' = h^{-1}(a'')$.

2) Let A^* be the set of homomorphisms $\chi : A \to \mathbb{C}^*$. Now H acts on A^* by

$$(h\chi)(a) = \chi(h^{-1}(a)), \quad h \in H,\ a \in A,\ \chi \in A^*$$

(We have $(h(h'(\chi)))(a) = h'(\chi)(h^{-1}(a)) = \chi(h'^{-1}h^{-1}(a)) = ((hh')\chi)(a)$.)

For any $\chi \in A^*$ let $H_\chi = \{h \in H \mid h\chi = \chi\}$.

For any $\chi \in A^*$ and any irr rep. ρ of H_χ we define a repres. ρ_χ of $H_\chi \times A$ by

$\rho_\chi(h, a) = \chi(a)\rho(h)$. We have

$$\rho_\chi((h,a)(h'a')) = \rho_\chi(hh', h'^{-1}(a) + a') \stackrel{?}{=} \rho_\chi(h,a)\rho_\chi(h', a')$$

$\iff$

$$\chi(h'^{-1}(a) + a')\rho(hh') \stackrel{?}{=} \chi(a)\rho(h)\chi(a')\rho(h')$$

$\iff$ $\chi(h'^{-1}(a)) \stackrel{?}{=} \chi(a)$ (holds since $h' \in H_\chi$).

We form $\tilde{\rho}_\chi = \mathrm{Ind}_{H_\chi \times A}^{G}(\rho_\chi)$ a repres. of G with character

$$f_{\chi,\rho}(h,a) = \frac{1}{\#(H_\chi \times A)} \sum_{\substack{(h'a') \in H \times A \\ (h_0, a_0) \in H_\chi \times A \\ (h'a')(h,a) = (h_0 a_0)(h', a')}} \chi(a_0)\, \mathrm{tr}(h_0, \rho)$$

3) The equation $(h' a')(h\ a) = (h_0\ a_0)(h' a')$ is

$$(h'h,\ h^{-1}(a') + a) = (h_0 h',\ h'^{-1}(a_0) + a') \quad \text{that is}$$

$$h_0 = h' h h'^{-1}, \quad a_0 = h' h^{-1}(a') + h'(a) - h'(a') \quad \text{and}$$

$$\chi(a_0) = \underbrace{\chi(h' h^{-1} a')}_{\substack{\chi(h_0^{-1} h' a') \\ = \chi(h'(a'))}} \chi(h'(a))\, \chi(h'(a'))^{-1} = \chi(h'(a))$$

Thus our character is

$$(h, a) \to \frac{1}{\#(H_\chi A)} \sum_{\substack{(h'\ a') \in H \times A \\ h' h h'^{-1} \in H_\chi}} \chi(h'(a))\, \mathrm{tr}(h' h h'^{-1}, \rho).$$

We compute the inner product (|) of this character with the analogous character attached to χ', ρ'. Using Frobenius reciprocity this is

$$\sum_{(h,a) \in H_{\chi'} \times A} \frac{1}{\#(H_\chi A)\ \#|H_{\chi'} A|} \sum_{\substack{(h, a') \in H \times A \\ h' h h'^{-1} \in H_\chi}} \chi(h'(a))\, \overline{\chi'(a)}\ \mathrm{tr}(h' h h'^{-1}, \rho)\ \overline{\mathrm{tr}(h, \rho')}$$

We have $\sum\limits_{a\in A} \chi(h'(a))\,\overline{\chi'(a)} =$

$$= \begin{cases} \#A & \text{if } h'^{-1}(\chi) = \chi' \\ 0 & \text{otherwise.} \end{cases}$$

Thus the entire sum is zero unless $\overbrace{\chi' = h'^{-1}(\chi)}^{\text{for some } h'\in H,}$ which we now assume. We obtain

$$\frac{\#(A)^2}{\#(H_\chi)^2\,\#(A)^2} \sum_{\substack{h'\in H \\ h'(\chi')=\chi}} \sum_{h\in H_{\chi'}} tr(h'hh'^{-1}, \rho)\,\overline{tr(h,\rho')}$$

We choose $h_0'\in H$ such that $h_0'(\chi') = \chi$ and we define a representation $\tilde{\rho}$ of $H_{\chi'}$ by $h \mapsto \rho(h_0' h h_0'^{-1})$. The sum becomes

$$\frac{1}{\#(H_\chi)^2}\,\#(H_\chi) \sum_{h\in H_{\chi'}} tr(h,\tilde{\rho})\,\overline{tr(h,\rho')}$$

$$= \begin{cases} 1 & \text{if } \tilde{\rho} \cong \rho' \\ 0 & \text{otherwise.} \end{cases}$$

We have thus a family of irreducible repres. of H indexed by (χ,ρ), where $\chi\in A^*$ is a representative of any H-orbit and ρ is an irred rep of H_χ (up to iso). The sum of squares of degrees of these representations is

$$\sum_{\substack{x \in A^* \\ \text{one in} \\ \text{each} \\ H\text{-orbit}}} \frac{\#(H \times A)^2}{\#(H_x \times A)^2} \sum_{\rho} \underbrace{\dim(\rho)^2}_{\# H_x} =$$

$$= \sum_{\substack{x \\ \text{one in} \\ \text{each } H\text{-orb}}} \frac{\#(H)^2}{\#(H_x)} = \# A \cdot \# H$$

Hence all irred. reps. of $H \times A$ are obtained.

The Weyl group of type B_n. $(n \geq 1)$

Let W_n be the group of all permutations of $\{1, 2, \ldots, n, n', \ldots, 2', 1'\}$ which commute with the involution $i \to i'$, $i' \to i$ $(1 \leq i \leq n)$. For each j, $1 \leq j \leq n-1$ let $s_j \in W_n$ be the permutation which interchanges j, $j+1$ and also j', $j'+1$ and leaves all other

elements unchanged. Let $s_n \in W_n$ be the permutation which interchanges n with n' and leaves the other elements unchanged. Then $S = \{s_1, s_2, \ldots, s_n\}$ generate W_n.
Let $\chi : W \to \pm 1$ be the homomorphism defined by $\chi(s_i) = 1$ $1 \leq i \leq n-1$, $\chi(s_n) = -1$. (Show that χ is well defined.)

A permutation in W_n defines a permutation of the n element set consisting of the unordered pairs $(1,1')$, $(2,2')$, $\ldots$, (n,n'). Thus we have a natural homomorphism of W_n onto the symmetric group S.
Let $r, \tilde{r}$ be integers ≥ 0 such that $r + \tilde{r} = n$. Let $W_{r,\tilde{r}}$ be the subgroup of W_n consisting of all permutations which map $\{1, 2, \ldots r, r', \ldots 2', 1'\}$ into itself and hence also map $\{r+1, \ldots, n, n', \ldots (r+1)'\}$ into itself. This can be regarded as $W_r \times W_{\tilde{r}}$ (convention: $W_0 = \{1\}$.).

7) Hence $W_{r,\tilde{r}}$ has a natural map (as above) onto the product $S_r \times S_{\tilde{r}}$ of two symmetric groups. Let E_1 be an irreducible rep. of S_r and let E_2 be an irred. rep. of $S_{\tilde{r}}$. We can regard $E_1 \otimes E_2$ as a $W_{r,\tilde{r}}$ representation via the projection $W_{r,\tilde{r}} \to S_r \times S_{\tilde{r}}$. We tensor $E_1 \otimes E_2$ with the 1-dimensional character of $H_{r,\tilde{r}}$ which is 1 on the W_r-factor and is the restriction of χ on the $W_{\tilde{r}}$-factor. We induce the resulting repres. from $W_{r,\tilde{r}}$ to W_n. We obtain thus an irreducible repres. of W_n. (In fact all irred. reps.).

This is a special case of a semidirect product. Let $A =$ all permutations in W_n which preserve each pair $(1,1'), (2,2') \ldots (n,n')$ an abelian group of order 2^n. Let H be the group of all permut in W_n which preserve $\{1,2,\ldots,n\}$ hence also $\{1',2',\ldots,n'\}$. Then W_n is the semidirect product of A, H with A normal.