1)

Some representations of $GL(V)$, V/F_q.

after Steinberg (1951)

Let $n \geq 1$. Let S be the group of all bijections $\sigma: I \xrightarrow{\sim} I$, $I = \{1, 2, \dots, n\}$.

Let $G = GL(V)$ be the group of all linear isomorphisms $V \xrightarrow{\sim} V$ where V is an n-dim vector space$/F_q$, a finite field with q elements.

Let Z'_n be the set of all sequences $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ of integers ≥ 0 with sum $= n$. Let $\nu \in Z'_n$.

Let X_ν be the set of all sequences of subsets

$I_1 \subset I_2 \subset \dots \subset I_n = I$ where

$\# I_1 = \nu_1$, $\# I_2 = \nu_1 + \nu_2$, $\dots$, $\# I_n = \nu_1 + \nu_2 + \dots + \nu_n = n$.

Then S acts transitively on X_ν by

$$\sigma : (I_1 \subset I_2 \subset \dots \subset I_n) \to (\sigma(I_1) \subset \sigma(I_2) \subset \dots \subset \sigma(I_n)).$$

Let $X_{\nu, q}$ be the set of sequences of subspaces $V_1 \subset V_2 \subset \dots \subset V_n = V$ where

$\dim V_1 = \nu_1$, $\dim V_2 = \nu_1 + \nu_2$, $\dots$, $\dim V_n = \nu_1 + \nu_2 + \dots + \nu_n = n$.

Then G acts transitively on $X_{\nu, q}$ by

$$s : (V_1 \subset V_2 \subset \dots \subset V_n) \to (s(V_1), s(V_2), \dots, s(V_n)).$$

2)

Hence the permutation representation $[X_\nu]$ of S and $[X_{\nu,q}]$ of G are defined.
Recall that Z_n is the set of all $\nu = (\nu_1, \dots \nu_n) \in Z_n'$ such that $\nu_1 \preceq \nu_2 \preceq \dots \preceq \nu_n$.

Recall that for $\nu \in Z_n$ we have

$$A_\nu = \sum_{\nu' \in Z_n} c_{\nu',\nu} [X_{\nu'}] \quad \text{in } \mathcal{R}(S)$$

where $c_{\nu',\nu}$ are uniquely defined integers. We define

$$A_{\nu,q} = \sum_{\nu' \in Z_n} c_{\nu',\nu} [X_{\nu',q}] \quad \text{in } \mathcal{R}(G)$$

<u>Theorem (Steinberg)</u>. 1) For any $\nu \in Z_n$, $A_{\nu,q}$ is $\pm$ irred. rep. of G.
2) For $\nu \neq \nu'$ in Z_n, $\pm A_{\nu,q}$, $\pm A_{\nu',q}$ are non-isomorphic.

The proof is based on the following

<u>Lemma</u>. For ν, ν' in Z_n, the number of G-orbits on $X_{\nu,q} \times X_{\nu',q}$ equals the number of S-orbits on $X_\nu \times X_{\nu'}$.

3)

We prove the theorem assuming the lemma. For ν', ν'' in Z_n we have

$$(\chi_{A_{\nu',q}} \mid \chi_{A_{\nu'',q}}) =$$

$$= \sum_{\mu',\mu''} c_{\mu',\nu'}\, c_{\mu'',\nu''} \cdot (\chi_{[X_{\mu',q}]} \mid \chi_{[X_{\mu'',q}]})$$

$$= \sum_{\mu',\mu''} c_{\mu',\nu'}\, c_{\mu'',\nu''} \,\#(G\text{-orbits on } X_{\mu',q} \times X_{\mu'',q})$$

$$= \sum_{\mu',\mu''} c_{\mu',\nu'}\, c_{\mu'',\nu''} \,\#(S\text{-orbits on } X_{\mu'} \times X_{\mu''})$$

$$= \sum_{\mu',\mu''} c_{\mu',\nu'}\, c_{\mu'',\nu''} \,(\chi_{[X_{\mu'}]} \mid \chi_{[X_{\mu''}]})$$

$$= (\chi_{A_{\nu'}} \mid \chi_{A_{\nu''}}) = \begin{cases} 1 & \text{if } \nu' = \nu'' \\ 0 & \text{if } \nu' = \nu''. \end{cases}$$

The theorem follows.

4)

We now prove the Lemma.

We write $\nu = (\nu_1, \nu_2, \dots, \nu_n)$, $\nu' = (\nu'_1, \nu'_2, \dots, \nu'_n)$.

Let S_ν be the group of all $\sigma \in S$ which leave stable each of the subsets

$$(*')\quad \{1, 2, \dots, \nu_1\},\ \{\nu_1+1, \nu_1+2, \dots, \nu_1+\nu_2\},$$
$$\{\nu_1+\nu_2+1, \nu_1+\nu_2+2, \dots, \nu_1+\nu_2+\nu_3\}, \dots$$

Let $S_{\nu'}$ be the group of all $\sigma \in S$ which leave stable each of the following subsets

$$(*'')\quad \{1, 2, \dots, \nu'_1\},\ \{\nu'_1+1, \nu'_1+2, \dots, \nu'_1+\nu'_2\},$$
$$\{\nu'_1+\nu'_2+1, \nu'_1+\nu'_2+2, \dots, \nu'_1+\nu'_2+\nu'_3\}, \dots$$

We can identify $X_\nu = S/S_\nu$, $X_{\nu'} = S/S_{\nu'}$
hence $\#(S \backslash (X_\nu \times X_{\nu'})) = \#(S_\nu \backslash S/S_{\nu'})$.

Let $V_* = (V_1 \subset V_2 \subset \dots \subset V_n) \in X_{\nu,q}$,
$V'_* = (V'_1 \subset V'_2 \subset \dots \subset V'_n) \in X_{\nu' q}$.

We can find a complete flag V^1_* in V and a complete flag $V^{1\prime}_*$ in V such that any V_i is one of the subspaces which form V^1_* and any V'_i is one of the subspaces which form $V^{1\prime}_*$.

5)

Let $\sigma^1 = pos(V_*^1, V_*^{1'}) \in S$. Now $V_*^1, V_*^{1'}$ are not necessarily unique. If $V_*^2, V_*^{2'}$ is another choice for $V_*^1, V_*^{1'}$ we set $\sigma^2 = pos(V_*^2, V_*^{2'}) \in S$. We show:

(a) $S_\nu \sigma^2 S_{\nu'} = S_\nu \sigma^1 S_{\nu'}$.

Let J (resp. J') be the set of all $i \in \{1, \dots, n-1\}$ such that $\sigma_i \in S_\nu$ (resp. $\sigma_i \in S_{\nu'}$). With an earlier notation we have $S_\nu = S_J$, $S_{\nu'} = S_{J'}$.
From the definition we have

$$z := pos(V_*^2, V_*^1) \in S_\nu = S_J, \; z' = pos(V_*^{1'}, V_*^{2'}) \in S_{\nu'} = S_{J'}.$$

and f_{σ^2} appears with $\neq 0$ coefficient in $T_z * T_{\sigma^1} * T_{z'}$ (product in $\mathcal{H}_q$).
From an earlier result, we see that $\sigma^2 \in S_J \sigma^1 S_{J'}$ and (a) follows. Thus $(V_*^1, V_*^{1'}) \to S_J\, pos(V_*^1, V_*^{1'})\, S_{J'}$ is a well defined map $X_{\nu, q} \times X_{\nu' q} \longrightarrow S_\nu \backslash S / S_{\nu'}$.

This map is obviously constant on G-orbits hence it induces a map

$$\begin{tikzcd} G\backslash(X_{\nu,q} \times X_{\nu',q}) \arrow[r, "\alpha"] & S_\nu\backslash S/S_{\nu'} \\ G\backslash(\mathcal{B} \times \mathcal{B}) \arrow[u, "\gamma"] \arrow[r, "\sim", "\delta"'] & S \arrow[u, "\beta"'] \end{tikzcd}$$

such that the obvious diagram (above) is commutative. We define $S \to G\backslash(X_{\nu,q} \times X_{\nu',q})$ as $\gamma\,\delta$. From the definitions this is constant on the $(S_\nu, S_{\nu'})$ double cosets hence it induces

$$\alpha' : S_\nu\backslash S/S_{\nu'} \longrightarrow G\backslash(X_{\nu,q} \times X_{\nu',q}).$$

From the definitions, α, α' are inverse to each other. Thus,

$$\#(G\backslash X_{\nu,q} \times X_{\nu',q}) = \#(S_\nu\backslash S/S_{\nu'}).$$

$\square$

If we assume Frobenius's theorem about the irr reps. of S we could deduce as before:

Let $A_{\nu,q} = \sum_{k} \operatorname{sgn}(k) \left[X_{\lambda-k,\, q} \right] \in R(G)$

k runs over all permutations of $\{0, 1, .., n-1\}$ such that $\lambda_i - k_i \geq 0 \ \forall i$.

Then

$\pm A_{\nu,q}$ is an irreducible repres of G.

Moreover the various $\pm A_{\nu,q}$ are distinct.

In fact, Steinberg shows that $\pm$ is $+$.

8)

Example. $(n=3)$

$[X_{1,1,1;q}] = \{$functions on $(V_1 \subset V_2)$ in V, $\dim V_i = i\}$

This décomposes as $M_1 \oplus M_2 \oplus M_3 \oplus M_4$

where

$$M_1 = \left\{ f \;\middle|\; \begin{array}{l} \sum\limits_{\substack{V_1 \\ V_1 \subset V_2}} f(V_1 \subset V_2) = 0, \ \forall V_2 \\ \sum\limits_{\substack{V_2 \\ V_1 \subset V_2}} f(V_1 \subset V_2) = 0, \ \forall V_1 \end{array} \right\}$$

$$M_2 = \left\{ f \;\middle|\; \sum_{V_1 \subset V_2} f(V_1 \subset V_2) = 0, \ f \text{ independent of } V_1 \right\}$$

$$M_3 = \left\{ f \;\middle|\; \sum_{V_1 \subset V_2} f(V_1 \subset V_2) = 0, \ f \text{ independent of } V_2 \right\}$$

$$M_4 = \{ f \mid \text{constant} \}.$$

We have $M_2 \cong M_3$ as irr repres. of dim q^2+q

M_1 irr rep. of dim q^3

M_4 " " " " 1

Note: $q^3 + (q^2+q) + (q^2+q) + 1 = (q+1)(q^2+q+1) = \# X_{111,q}$