

1) Hecke algebra.

Let V be a vector space of dim n over F_2 , a finite field.

A complete flag in V is a sequence $V_* = (V_0 \subset V_1 \subset \dots \subset V_n)$

of subspaces of V , $\dim V_i = i$, $\forall i$. Let $B = \text{set of complete flags}$. For

$V_*, V'_* = (V'_0 \subset \dots \subset V'_n)$ in B we define a permutation of $\{1, 2, \dots, n\}$. Let $i \in \{1, 2, \dots, n\}$. The intersections

$$V_0 \cap V'_i \subset V_1 \cap V'_i \subset V_2 \cap V'_i \subset \dots \subset V_n \cap V'_i$$

have dimensions starting with 0, ending with i and at each step they increase by 1 or remain the same. So there are exactly i places

where the dimension increases by 1. Of these, $i-1$ places are obtained from $V'_1, V'_2, \dots, V'_{i-1}$.

We denote the new place by $\sigma(i)$. It is

different from $\sigma(1), \dots, \sigma(i-1)$. This defines the

permutation $\sigma = \text{pos}(V_*, V'_*)$, the relative position of

V_*, V'_* . From definitions we have

$$(*) \quad V_{\sigma(i)-1} \cap V'_i \subset V_{\sigma(i)} \cap V'_i$$

for $i=1, \dots, n$. We can find a line $\tilde{L}_i$

such that

$$V_{\sigma(i)} \cap V'_i = (V_{\sigma(i)-1} \cap V'_i) \oplus \tilde{L}_i$$

for $i=1, \dots, n$.

2)

Then we have also

$$(*) V_j \cap V'_{\sigma^{-1}(j)} = (V_{j-1} \cap V'_{\sigma^{-1}(j)}) \oplus \tilde{L}_{\sigma^{-1}(j)}.$$

for $j=1, \dots, n$.

From $(*)$ we have $\tilde{L}_{\sigma^{-1}(j)} \subset V_j$, $\tilde{L}_{\sigma^{-1}(j-1)} \subset V_{j-1} \subset V_j$,
 $\dots$, $\tilde{L}_{\sigma^{-1}(1)} \subset V_1 \subset V_j$ hence $\tilde{L}_{\sigma^{-1}(j)} + \tilde{L}_{\sigma^{-1}(j-1)} + \dots + \tilde{L}_{\sigma^{-1}(1)}$

is contained in V_j . We show that this sum is direct for $j \geq 1$. By induction on j . If $j=1$ this is obvious. Assume $j \geq 2$. By the induction hypothesis $V_{j-1} = \tilde{L}_{\sigma^{-1}(j-1)} \oplus \dots \oplus \tilde{L}_{\sigma^{-1}(1)}$. It is enough to show that

$\tilde{L}_{\sigma^{-1}(j)} \not\subset V_{j-1}$. If $\tilde{L}_{\sigma^{-1}(j)} \subset V_{j-1}$ then the $\tilde{L}_{\sigma^{-1}(j)} \subset V_{j-1} \cap V'_{\sigma^{-1}(j)}$ contradicting $(*)$.

This proves:

$$V_j = \tilde{L}_{\sigma^{-1}(j)} \oplus \tilde{L}_{\sigma^{-1}(j-1)} \oplus \dots \oplus \tilde{L}_{\sigma^{-1}(1)} \quad \forall j$$

In particular $\tilde{L}_1 \oplus \tilde{L}_2 \oplus \dots \oplus \tilde{L}_n = \tilde{V}$ is a direct sum.

From $(*)$ we have $\tilde{L}_i \subset V'_i$, $\tilde{L}_{i-1} \subset V'_{i-1} \subset V'_i$, $\dots$, $\tilde{L}_1 \subset V'_i$.
 hence $V'_i = \tilde{L}_1 \oplus \tilde{L}_2 \oplus \dots \oplus \tilde{L}_i$. Set $L_i = \tilde{L}_{\sigma^{-1}(i)}$. Then
 $V_j = L_1 \oplus \dots \oplus L_j$, $V'_i = L_{\sigma(1)} \oplus L_{\sigma(2)} \oplus \dots \oplus L_{\sigma(i)}$.

3) Let $V_*^0 = (V_1^0 \subset \dots \subset V_n^0)$, $V_*^1 = (V_1^1 \subset \dots \subset V_n^1)$, $\tilde{V}_*^0 = (\tilde{V}_1^0 \subset \dots \subset \tilde{V}_n^0)$,
 $\tilde{V}_*^1 = (\tilde{V}_1^1 \subset \dots \subset \tilde{V}_n^1)$ be in $\mathcal{B}$ and let

$$\sigma^0 = \text{pos}(V_*^0, \tilde{V}_*^0), \quad \sigma^1 = \text{pos}(V_*^1, \tilde{V}_*^1).$$

Lemma. We have (1) $\sigma^0 = \sigma^1$ if and only if
 we have (2) $s(V_*^0) = V_*^1$, $s(\tilde{V}_*^0) = \tilde{V}_*^1$ for some $g \in G$.

Proof. (2) $\Rightarrow$ (1) is immediate from definitions.

Assume now that $\sigma^0 = \sigma^1 = \sigma$.

We can find lines $L_1^0, \dots, L_n^0$ and
 lines $L_1^1, \dots, L_n^1$ such that for any i :

$$V_i^0 = L_1^0 \oplus \dots \oplus L_i^0, V_i^1 = L_1^1 \oplus \dots \oplus L_i^1,$$

$$\tilde{V}_i^0 = L_{\sigma(1)}^0 \oplus \dots \oplus L_{\sigma(i)}^0,$$

$$\tilde{V}_i^1 = L_{\sigma(1)}^1 \oplus \dots \oplus L_{\sigma(i)}^1.$$

In particular $V = L_1^0 \oplus \dots \oplus L_n^0 = L_1^1 \oplus \dots \oplus L_n^1$.

We can find $s \in G$ such that
 $s(L_i^0) = L_i^1$ for all i . Then $s(\tilde{V}_i^0) = \tilde{V}_i^1$ for
 all i and $s(V_j^0) = V_j^1$ for all j . Thus (2)
 holds. $\square$

4)

For any $\sigma \in S$ let

$$\mathcal{O}_\sigma = \{ (V_x, V'_x) \in B \times B \mid \text{pos}(V_x, V'_x) = \sigma \}.$$

This set is

non-empty: we can find lines

$L_1, \dots, L_n$ in V such that $V = L_1 \oplus \dots \oplus L_n$.

We define $V_x = (V_1 \subset \dots \subset V_n)$ by

$$V_i = L_1 \oplus \dots \oplus L_i. \quad \text{We define}$$

$$V'_x = (V'_1 \subset \dots \subset V'_n) \text{ by } V'_i = L_{\sigma(1)} \oplus \dots \oplus L_{\sigma(i)}$$

$$\text{Then } \text{pos}(V_x, V'_x) = \sigma.$$

Also $\mathcal{O}_\sigma$ is a single G -orbit on $B \times B$.

Thus,

$$\{ G\text{-orbits on } B \times B \} \xrightarrow[\text{pos}]{\sim} S'$$

This is called Bruhat decomposition.

First stated in this form by R. Steinberg (1951), Gelfand-Naimark (1950) and in a somewhat less precise form by C. Ehresmann (1935).

5) We choose a basis $\{e_1, \dots, e_n\}$ of V . Any $s \in G$ can be written as a matrix (s_{ij}) where $s e_j = \sum s_{ij} e_i$. We identify $\sigma \in S$ with the element of G given by $\sigma e_j = e_{\sigma(j)}$. We set $V_\sigma = (V_0 \subset V_\sigma \subset \dots \subset V_n)$ where $V_0 = 0$, $V_1 = \text{span}(e_1)$, $V_2 = \text{span}(e_1, e_2)$, etc.

We have $V_\sigma \in \mathcal{B}$.

We show:

(*) $\text{pos}(V_\sigma, \sigma(V_\sigma)) = \sigma$.
 Let $L_i = \text{span}(e_i)$. Let $V'_\sigma = \sigma(V_\sigma)$
 $V'_i = L_1 + \dots + L_i$
 $V'_i = L_{\sigma(1)} + \dots + L_{\sigma(i)}$

hence (*) holds.

Lemma Let $V_\sigma = (V_0 \subset \dots \subset V_n)$, $V'_\sigma = (V'_0 \subset \dots \subset V'_n)$ be in $\mathcal{B}$, $\sigma = \text{pos}(V_\sigma, V'_\sigma)$. Then $\text{pos}(V'_\sigma, V_\sigma) = \sigma^{-1}$.
 Proof. Consider lines $L_1, \dots, L_n$ with $V_i = L_1 \oplus \dots \oplus L_i$, $V'_i = L_{\sigma(1)} \oplus \dots \oplus L_{\sigma(i)}$.
 Set $L'_i = L_{\sigma(i)}$. Then $V'_i = L'_1 + \dots + L'_i$, $V_i = L_{\sigma^{-1}(1)} \oplus \dots \oplus L_{\sigma^{-1}(i)}$. $\square$

6)

Consider

$V_* = (V_0 \subset \dots \subset V_n)$, $V'_* = (V'_0 \subset \dots \subset V'_n)$,
 $V''_* = (V''_0 \subset \dots \subset V''_n)$ in $\mathcal{B}$. Let $\sigma = \text{pos}(V_*, V'_*)$,
 $1 \leq i \leq n$, $\sigma(i) < \sigma(i+1)$, $\tau = \text{pos}(V_*, V''_*)$,
 $\sigma_i = \text{pos}(V'_*, V''_*)$, where $\sigma_i: i \rightarrow i+1, i+1 \rightarrow i, j \rightarrow j$ for
 $j \neq i, i+1$

Lemma $\tau = \sigma \sigma_i$.

Proof. We have $V''_j = V'_j$ for $j \neq i$, $V''_i \neq V'_i$.

It is enough to show: (A) $V_{\sigma \sigma_i(j)-1} \cap V''_j \subset V_{\sigma \sigma_i(j)} \cap V''_j$
for all j . When $j \neq i, i+1$, (A) follows from

$V_{\sigma(j)-1} \cap V'_j \subset V_{\sigma(j)} \cap V'_j$. If $j = i+1$, (A) follows from
 $V_{\sigma(i)-1} \cap V'_{i+1} \subset V_{\sigma(i)} \cap V'_{i+1}$ which is known from

the construction of σ . It remains to prove (A) for
 $j = i$. Assume it is not true, that is,

(*) $V_{\sigma(i+1)-1} \cap V''_i = V_{\sigma(i+1)} \cap V''_i$.

We have

A	$\subset$	B	, where	$A = V_{\sigma(i+1)-1} \cap V'_{i+1}$
$\cup$		$\cup$		
C	$=$	D		$B = V_{\sigma(i+1)} \cap V'_{i+1}$
$\cup$		$\cup$		
E	$\subset$	F		$C = V_{\sigma(i+1)-1} \cap V''_i$
				$D = V_{\sigma(i+1)} \cap V''_i$
				$E = V_{\sigma(i+1)-1} \cap V'_{i-1}$
				$F = V_{\sigma(i+1)} \cap V'_i$

Let $L_1, \dots, L_n$ be lines such that $V_i = L_1 \oplus \dots \oplus L_i$, (7)

$V'_j = L_{\sigma(1)} \oplus \dots \oplus L_{\sigma(j)}$, $\forall j$. We have

$$A = \bigoplus \{ L_{\sigma(j)} ; j \leq i+1, \sigma(j) \leq \sigma(i+1)-1 \}$$

$$B = \bigoplus \{ L_{\sigma(j)} ; j \leq i+1, \sigma(j) \leq \sigma(i+1) \}$$

$$E = \bigoplus \{ L_{\sigma(j)} ; j \leq i-1, \sigma(j) \leq \sigma(i+1)-1 \}$$

$$F = \bigoplus \{ L_{\sigma(j)} ; j \leq i-1, \sigma(j) \leq \sigma(i+1) \}.$$

In A we must have $j \neq i+1$, and $j=i$ appears in A (assumption $\sigma(i) < \sigma(i+1)$), not in E . Thus $\dim(A/E) = 1$

In B both $j=i, i+1$ appear, not in F . Thus $\dim(B/F) = 2$

We have $\dim(B/A) = 1$ by definition of σ . We have

$E = F$. [If $j \leq i-1$, $\sigma(j) \leq \sigma(i+1)$ then $\sigma(j) < \sigma(i+1)$

$\Rightarrow j \neq i+1$. We have $\dim B/D \leq 1$ $\dim D/F \leq 1$

since $\dim V'_{i+1}/V''_i = 1$, $\dim V''_i/V'_{i-1} = 1$. Since

$\dim B/F = 2$, it follows that $\dim B/D = \dim D/F = 1$

Thus, $D \neq F$. Since $C = D$, $E = F$, we have $C \neq E$. Since

$\dim(A/E) = 1$, $E \subset C \subset A$, it follows that $C = A$.

Since $L_{\sigma(i)} \subset A$, it follows that $L_{\sigma(i)} \in C$ hence

$L_{\sigma(i)} \in V''_i$. We have also $L_{\sigma(1)}, \dots, L_{\sigma(i-1)} \subset V''_i$ since

they are in V'_i . Thus $V''_i = L_{\sigma(1)} + \dots + L_{\sigma(i-1)} + L_{\sigma(i)} = V'_i$,

contradiction. $\square$

We state a converse of the lemma above. (8)

Consider $V_x = (V_0 \dots V_n)$, $V_x'' = (V_0'' \dots V_n'')$ in $\mathcal{B}$ such that $\text{pos}(V_x, V_x'') = \sigma \sigma_i$, with σ, i, σ_i as before.

Lemma. There is a unique $V_x' = (V_0' \dots V_n') \in \mathcal{B}$ such that $\text{pos}(V_x, V_x') = \sigma$, $\text{pos}(V_x', V_x'') = \sigma_i$.

Proof. We set $V_j' = V_j''$ for $j \neq i$. We define A, B, E, F as in the previous lemma. We have $\dim(B/A) = 1$, $\dim(A/E) = 1$, $\dim(B/F) = 2$, $E = F$.

Now V_i' can be viewed as a line in the plane $V_{i+1}'/V_{i-1}' = B/F$; in fact it is the line $A/F \subset B/F$. $\square$

Let $\mathcal{H} = \left\{ f : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{C} \mid f \text{ constant on the } G\text{-orbits on } \mathcal{B} \times \mathcal{B} \right\}$

Here G acts on $\mathcal{B} \times \mathcal{B}$ by $s : (V_x, V_x') \rightarrow (s(V_x), s(V_x'))$

For any $\sigma \in S$ let $T_\sigma : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{C}$ be the function

$$\begin{aligned} (V_x, V_x') &\rightarrow 1 \text{ if } (V_x, V_x') \in \mathcal{O}_\sigma \\ (V_x, V_x') &\rightarrow 0 \text{ if } (V_x, V_x') \notin \mathcal{O}_\sigma. \end{aligned}$$

Then $\{T_\sigma; \sigma \in S\}$ is a $\mathbb{C}$ -basis of $\mathcal{H}$.

(9)

For T, T' in $\mathcal{H}$ we define $T * T' : B \times B \rightarrow \mathbb{C}$

by $(T * T')(v_*, v_*'') = \sum_{\substack{v_*' \in B}} T(v_*, v_*') T'(v_*', v_*'')$.

We have $T * T' \in \mathcal{H}$. this defines
an associative algebra structure on $\mathcal{H}$ with unit T_1 .

For $\sigma, \sigma' \in S$ we have

$$T_\sigma * T_{\sigma'} = \sum_{\sigma'' \in S} c_{\sigma \sigma'}^{\sigma''} T_{\sigma''}$$

where

$$c_{\sigma \sigma'}^{\sigma''} = \# \{ v_*' \in B \mid (v_*, v_*') \in \mathcal{O}_\sigma, (v_*', v_*'') \in \mathcal{O}_{\sigma'} \}$$

for fixed $(v_*, v_*'') \in \mathcal{O}_{\sigma''}$. (This is an integer independent of the choice of (v_*, v_*'') .)

The algebra $\mathcal{H}$ is called the Hecke algebra. It was introduced by Iwahori (1965).

We describe the multiplication $*$ more explicitly.

For $\sigma \in S$ we set

$$I(\sigma) = \left\{ \binom{i}{j} \mid \begin{array}{l} i, j \text{ in } \{1, \dots, n\} \times \{1, \dots, n\} \\ i < j, \sigma(i) > \sigma(j) \end{array} \right\}, \quad |\sigma| = \# [I(\sigma)]$$

(length function).

For $i \in \{1, \dots, n-1\}$ define $\sigma_i \in S$ by

$$i \rightarrow i+1, i+1 \rightarrow i, j \rightarrow j \text{ for } j \neq i$$

10) For $\sigma \in \mathcal{O}$ we have $|\sigma \sigma_i| = |\sigma| + 1$ if $\sigma(i) < \sigma(i+1)$
 (Exercise.) $|\sigma| - 1$ if $\sigma(i) > \sigma(i+1)$.

Lemma. If $|\sigma \sigma_i| = |\sigma| + 1$, then $T_\sigma * T_{\sigma_i} = T_{\sigma \sigma_i}$.

Proof. We use that if $(v_*, v'_*) \in \mathcal{O}_\sigma$, $(v'_*, v''_*) \in \mathcal{O}_{\sigma_i}$ then $(v_*, v''_*) \in \mathcal{O}_{\sigma \sigma_i}$ and for $(v_*, v''_*) \in \mathcal{O}_{\sigma \sigma_i}$ there is a unique $v'_* \in \mathcal{B}$ such that $(v_*, v'_*) \in \mathcal{O}_\sigma$, $(v'_*, v''_*) \in \mathcal{O}_{\sigma_i}$.

Lemma. We have $T_{\sigma_i}^2 = (q-1)T_{\sigma_i} + qT_1$.

Proof. Let $(v_*, v'_*) \in \mathcal{O}_{\sigma_i}$, $(v'_*, v''_*) \in \mathcal{O}_{\sigma_i}$. We have $v_j = v'_j = v''_j$ for $j \neq i$. If $v_i = v''_i$ then $(v_*, v'_*) \in \mathcal{O}_1$; if $v_i \neq v''_i$ then $(v_*, v''_*) \in \mathcal{O}_{\sigma_i}$. Thus $\mathcal{L}_{\sigma_i}^2 = c\mathcal{L}_{\sigma_i} + d\mathcal{L}_1$ for some c, d in $\mathbb{F}$. Assume that $(v_*, v''_*) \in \mathcal{O}_{\sigma_i}$.

$$\begin{aligned} & \# \{v'_* \in \mathcal{B} \mid (v_*, v'_*) \in \mathcal{O}_{\sigma_i}, (v'_*, v''_*) \in \mathcal{O}_{\sigma_i}\} \\ &= \# \{v'_i \mid v_{i-1} < v'_i < v_{i+1}, v'_i \neq v_i, v'_i \neq v''_i\} \\ &= \# [\text{lines in a 2-dim. vector space}] - 2 = q-1 \end{aligned}$$

Assume that $(v_*, v'_*) \in \mathcal{O}_1$. Then

$$\begin{aligned} & \# \{v'_* \in \mathcal{B} \mid (v_*, v'_*) \in \mathcal{O}_{\sigma_i}, (v'_*, v''_*) \in \mathcal{O}_{\sigma_i}\} \\ &= \# \{v'_i \mid v_{i-1} < v'_i < v_{i+1}, v'_i \neq v_i\} \\ &= \# [\text{lines in a 2-dim vector space}] - 1 = q. \quad \square \end{aligned}$$

Lemma. If $|\sigma_i \sigma| = |\sigma| + 1$ then $T_{\sigma_i} * T_\sigma = T_{\sigma_i \sigma}$

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Proof. We have a vector space isomorphism $\Theta: \mathcal{H} \rightarrow \mathcal{H}$ given by $(\oplus T)(v_x, v'_x) = T(v'_x, v_x)$.

It clearly satisfies $\Theta(Tx') = \Theta(T') \oplus \Theta(T)$.

It also satisfies $\oplus(T_\sigma) = T_{\sigma^{-1}}$ for $\sigma \in S$.

From $|\sigma_i \sigma| = |\sigma| + 1$ we have

$$|(\sigma_i \sigma)^{-1}| = |\sigma^{-1}| + 1 \quad (\text{see below}) \quad \text{hence } |\sigma^{-1} \sigma_i| = |\sigma^{-1}| + 1$$

Using the previous lemma we have

$$\oplus(T_{\sigma^{-1} \sigma_i}) = \oplus(T_{\sigma_i}) \oplus \oplus(T_{\sigma^{-1}}) \quad \text{hence}$$

$$T_{\sigma_i \sigma} = T_{\sigma_i} T_\sigma. \quad \square.$$

It remains to show that for any $\sigma \in S$ we have $|\sigma^{-1}| = |\sigma|$.

$$\begin{aligned} \text{We have } I_\sigma &= \{(i, j) \mid i < j, \sigma(i) > \sigma(j)\} \\ I_{\sigma^{-1}} &= \{(i', j') \mid i' < j', \sigma^{-1}(i') > \sigma^{-1}(j')\} \end{aligned}$$

Now we have a bijection $I_\sigma \xrightarrow{\sim} I_{\sigma^{-1}}$
 $(i, j) \rightarrow (i', j')$ where $i' = \sigma(j)$, $j' = \sigma(i)$.
 Thus $|\sigma^{-1}| = |\sigma|$.

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For any $\sigma \in S$, $\sigma \neq 1$, we can find $i \in \{1, \dots, n-1\}$ such that $\sigma(i) > \sigma(i+1)$. Otherwise we have $\sigma(i) < \sigma(i+1)$ for $i=1, \dots, n-1$ then $\sigma(i) < \sigma(j)$ for all $i < j$ hence $\sigma = 1$.

It follows that

(a) Any $\sigma \in S$ is of the form $\sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_k}$ for $i_1, i_2, \dots, i_k$ in $\{1, \dots, n-1\}$ with $k = |\sigma|$.

We use induction on $|\sigma|$. If $|\sigma| = 0$ then $\sigma = 1$ and (a) is obvious. Assume now that $|\sigma| > 1$. As we have seen we can find $i \in \{1, \dots, n-1\}$ such that $|\sigma \sigma_i| < |\sigma|$. Set $\sigma' = |\sigma \sigma_i|$. Then $\sigma = \sigma' \sigma_i$ and the induction hypothesis applies to σ' . $\square$

conversely, if $\sigma = \sigma_{i_1} \dots \sigma_{i_k}$ with $i_1, \dots, i_k$ in $\{1, \dots, n-1\}$ then $|\sigma| \leq k$. Indeed let $\sigma' = \sigma_{i_1} \dots \sigma_{i_{k-1}}$. Then by ind. $|\sigma'| \leq k-1$. We have $|\sigma| = |\sigma' \sigma_{i_k}| \leq |\sigma'| + 1 = k$. $\square$.

We show: if σ, σ' are in S then $|\sigma \sigma'| \leq |\sigma| + |\sigma'|$. Indeed, can write $\sigma = \sigma_{i_1} \dots \sigma_{i_k}$, $|\sigma| = k$ and $\sigma' = \sigma_{j_1} \dots \sigma_{j_l}$, $|\sigma'| = l$. Then $\sigma \sigma' = \sigma_{i_1} \dots \sigma_{i_k} \sigma_{j_1} \dots \sigma_{j_l}$ hence $|\sigma \sigma'| \leq k + l = |\sigma| + |\sigma'|$.

13)

Let J, J' be two subsets of $\{1, \dots, n-1\}$.

Let S_J (resp. $S_{J'}$) be the subgroup of S generated by $\{s_i; i \in J\}$ (resp. $\{s_i; i \in J'\}$).

Lemma. Let $\sigma \in S$, $\tau \in S_J$, $\tau' \in S_{J'}$. Then $T_\tau * T_\sigma * T_{\tau'}$ is a linear combination of elements $T_{\sigma'}$ with $\sigma' \in S_J \circ S_{J'}$.

Proof. Let $k = |\tau|$, $k' = |\tau'|$. We argue by induction on $k+k'$. If $k+k' = 0$ then $\tau = \tau' = 1$ and there is nothing to prove. Assume now that $k+k' \geq 1$. Then either $\tau \neq 1$ or $\tau' \neq 1$.

Assume first that $\tau' \neq 1$. We can find $i \in \{1, \dots, n-1\}$ such that $\tau'(i) > \tau'(i+1)$. We have

necessarily $i \in J'$ and $|\tau' \sigma_i| < |\tau'|$ hence
 $T_\tau * T_\sigma * T_{\tau'} = T_\tau * T_\sigma * T_{\tau' \sigma_i} * T_{\sigma_i} = \text{comb. of}$
 $T_{\sigma'} * T_{\sigma_i}, \quad \sigma' \in S_J \circ S_{J'}, i \in J'.$

Similarly if $\tau \neq 1$ we have that $T_\tau * T_\sigma * T_{\tau'}$ is a lin. com. of $T_{\sigma_i} * T_{\sigma'}, \quad \sigma' \in S_J \circ S_{J'}, i \in J.$

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Thus we are reduced to the case where

(1) $\tau = 1$, $\tau' = \sigma_i$ ($i \in J'$) or (2) $\tau = \sigma_i$ ($i \in J$), $\tau' = 1$.

In case (1), if $|\sigma\sigma_i| > |\sigma|$ then

$T_\sigma * T_{\sigma_i} = T_{\sigma\sigma_i}$ as required. If $|\sigma\sigma_i| < |\sigma|$

then setting $\tilde{\sigma} = \sigma\sigma_i$ we have $|\tilde{\sigma}\sigma_i| > |\tilde{\sigma}|$ and

$T_\sigma * T_{\sigma_i} = T_{\tilde{\sigma}} * T_{\sigma_i} * T_{\sigma_i} = \text{lin. comb. of}$

$T_{\tilde{\sigma}} * T_{\sigma_i} = T_\sigma$ and $T_{\tilde{\sigma}} = T_{\sigma\sigma_i}$ as required.

Case (2) is similar. $\square$.

We show:

If $\sigma, \sigma' \in S$ satisfy $|\sigma\sigma'| = |\sigma| + |\sigma'|$ then
 $T_\sigma * T_{\sigma'} = T_{\sigma\sigma'}$.

Induction on $|\sigma'|$. If $|\sigma'| = 0$ then $\sigma' = 1$ and result is clear. Assume $|\sigma'| \geq 1$. Then can find $i \in \{1, \dots, n-1\}$

such that $\sigma' = \sigma''\sigma_i$, $|\sigma'| = |\sigma''| + 1$. Assume that
 $|\sigma\sigma'| = |\sigma\sigma''| - 1$. Then $|\sigma\sigma'| \leq |\sigma| + |\sigma''| - 1 = |\sigma| + |\sigma'| - 2$
 contradiction. Thus $|\sigma\sigma'| = |\sigma\sigma''\sigma_i| + 1$ and

$T_{\sigma\sigma'} = T_{\sigma\sigma''} * T_{\sigma_i} = (\text{ind. hyp.}) = T_\sigma * T_{\sigma''} * T_{\sigma_i} = T_\sigma * T_{\sigma'}$

$\square$

15)

It follows that:

If $(V_x, V_x'') \in B \times B$ satisfy $(V_x, V_x'') \in \mathcal{O}_{\sigma\sigma'}$ with $|\sigma\sigma'| = |\sigma| + |\sigma'|$ then there is a unique $V_x' \in B$ such that $(V_x, V_x') \in \mathcal{O}_\sigma$, $(V_x', V_x'') \in \mathcal{O}_{\sigma'}$.