1)

Representations of the symmetric group

Let $I = \{1, 2, \dots, n\}$. Let S be the group of all bijections $\sigma : I \xrightarrow{\sim} I$. Let Z'_n (resp. Z_n) be the set of all sequences $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ (resp. $\nu = (\nu_1 \leq \nu_2 \leq \dots \leq \nu_n)$) of integers ≥ 0 with $\nu_1 + \nu_2 + \dots + \nu_n = n$.

For $\nu \in Z'_n$ we set $|\nu| = \binom{\nu_1}{2} + \binom{\nu_2}{2} + \dots + \binom{\nu_n}{2} \in \mathbb{Z}_{\geq 0}$;

$\Xi_\nu = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} ;\ i < j,\ \{i,j\} \subset$ one of:

$(*_\nu)$ $\{1, 2, \dots, n_1\}, \{n_1+1, n_1+2, \dots, n_1+n_2\}, \{n_1+n_2+1, \dots, n_1+n_2+n_3\}, \dots$ Let $\mathcal{P} = \mathbb{C}[x_1, \dots, x_n]$, x_i indeterminates; a $\mathbb{C}$-vector space, $\mathcal{P} = \bigoplus_{r \geq 0} \mathcal{P}_r$, $\mathcal{P}_r$ = polynomials of degree n.

For any $\nu \in Z'_n$ we set

$$\varphi_\nu = \prod_{(i,j) \in \Xi_\nu} (x_i - x_j) \in \mathcal{P}_{|\nu|}$$

Define a linear repres. $\rho : S \to GL(\mathcal{P}_{|\nu|})$ by

$$\rho(\sigma) = [f(x_1, \dots, x_n) \to f(x_{\sigma(1)}, \dots, x_{\sigma(n)})]$$

for $\sigma \in S$.

2)

<u>Proposition</u>. Let $\nu \in \mathbb{Z}_n'$.
Let A_ν be the subspace of $\mathcal{P}_{|\nu|}$ spanned by $\{\rho(\sigma)(\varphi_\nu) \; ; \; \sigma \in S\}$. This is an irreducible subrepresentation of $\mathcal{P}_{|\nu|}$

<u>Proof</u>. Clearly $\dim A_\nu < \infty$ and A_ν is S-stable.

Let $\theta : A_\nu \to A_\nu$ be a linear map commuting with the S-action.

Let $S_\nu = \{\sigma \in S \mid \sigma$ preserves each of the subsets $X_\nu\}$

Let $\varepsilon : S \to \{\pm 1\}$ be the sign homomorphism. If $\sigma \in S_\nu$, then $\rho(\sigma)(\varphi_\nu) = \varepsilon(\sigma)\varphi_\nu$. Hence
$\rho(\sigma)(\theta(\varphi_\nu)) = \varepsilon(\sigma)\, \theta(\varphi_\nu)$.

3) If $\Sigma \in \mathcal{P}_{|\nu|}$ satisfies $\rho(\sigma)\Sigma = \varepsilon(\sigma)\Sigma$ for $\sigma \in S_\nu$, then Σ must vanish whenever we set $x_i = x_j$ $\left((i\,j) \in \Xi_\nu\right)$, so that Σ is divisible by such $x_i - x_j$ hence also by $\prod_{(i,j)\in\Xi_\nu}(x_i - x_j) = \varphi_\nu$. Thus $\Theta(\varphi_\nu)$ is

divisible by φ_ν. Since $\Theta(\varphi_\nu)$, φ_ν are both in $\mathcal{P}_{|\nu|}$, we must have $\Theta(\varphi_\nu) = c\,\varphi_\nu$, c = constant.

It follows that $\Theta(\rho(\sigma)\varphi_\nu) =$

$= \rho(\sigma)(\Theta(\varphi_\nu)) = \rho(\sigma)(c\,\varphi_\nu) = c\,\rho(\sigma)\varphi_\nu$, $\forall \sigma \in S$.

Thus $\Theta = c \cdot 1$ on A_ν. We see that $\dim(\rho;\rho) = 1$ and

A_ν is irreducible.

4) **Lemma.** Let ν, ν' in Z_n be such that
$A_\nu, A_{\nu'}$ are isomorphic as repres. of S. Then $A_\nu \cap A_{\nu'} \neq 0$

Proof. Let $\Theta : A_\nu \to A_{\nu'}$ be an isomorphism commuting with action of S. As in the proof of the previous result, we see that $\Theta(\varphi_\nu)$ is divisible by φ_ν. Now $\Theta(\varphi_\nu) \in \mathcal{P}_{|\nu'|}$. Hence $|\nu'| \geq |\nu|$.

Reversing the roles of ν, ν' we see that $|\nu| \geq |\nu'|$ hence $|\nu| = |\nu'|$. We also see

that $\Theta(\varphi_\nu) = c\varphi_\nu$ for some scalar $c \neq 0$. Thus $\varphi_\nu \in A_{\nu'}$. We see that $A_\nu \cap A_{\nu'} \neq 0$ (it contains φ_ν).

5/ <u>Lemma.</u> Let ν, ν' in Z_n be such that $A_\nu \cap A_{\nu'} \neq \emptyset$. Then $\nu = \nu'$.

<u>Proof.</u> Let $\nu = (\nu_1 \leq \nu_2 \leq \dots \leq \nu_n)$, $\nu' = (\nu'_1 \leq \nu'_2 \leq \dots \leq \nu'_n)$.

Now φ_ν is a product of vandermonde determinants.
Hence $\varphi_\nu \in E$ where E is the subspace of $\mathcal{P}_{|\nu|}$ spanned by the monomials

$$x_1^{c_1} x_2^{c_2} \dots x_n^{c_n}$$ where

$(c_1, c_2, \dots, c_n)$ is a permutation of

$$(0, 1, 2, \dots, \nu_1 - 1, 0, 1, 2, \dots \nu_2 - 1, \dots, 0, 1, 2, \dots, \nu_n - 1).$$

Hence $A_\nu \subset E$. Similarly, $A_{\nu'} \subset E'$ where E' is the subspace of $\mathcal{P}_{|\nu'|}$ spanned by the monomials $x_1^{c'_1} x_2^{c'_2} \dots x_n^{c'_n}$ where $(c'_1, c'_2, \dots, c'_n)$ is a permutation of

$$(0, 1, 2, \dots, \nu'_1 - 1, 0, 1, 2, \dots, \nu'_2 - 1, \dots, 0, 1, 2, \dots, \nu'_n - 1).$$

6)

Since $A_\nu \cap A_{\nu'} \neq 0$ we have $|\nu| = |\nu'|$, $E \cap E' \neq 0$ hence

(*) $(0, 1, \dots, \nu_1 - 1, 0, 1, \dots, \nu_2 - 1, \dots, 0, 1, \dots, \nu_n - 1)$

is a permutation ~~of~~

$(0, 1, \dots, \nu_1' - 1, 0, 1, \dots, \nu_2' - 1, \dots, 0, 1, \dots, \nu_n' - 1)$.

It is enough to show that $(\nu_1, \nu_2, \dots, \nu_n)$ can be reconstructed from (*) or equivalently, from the unordered collection:

(**) $(1, 2, \dots, \nu_1, 1, 2, \dots \nu_2, \dots, 1, 2, \dots, \nu_n)$.

Let $M = \max(**)$. It appears in (**) say l_M times. The largest number in ν is also M and it appears there l_M times. We remove from (**) the entries $1, 2, \dots, \nu_i$ for any i s.t. $\nu_i = M$. The resulting sequence is $(**)'$; let $M' = \max(**')$; it appears in $(**)'$ $l_{M'}$ times; M' is also the largest number in ν from which all M are removed and it appears there $l_{M'}$ times. We continue and

7) see that all numbers in ν can be reconstructed from $\#1$. $\square$.

<u>Lemma</u>. Let ν, ν' in Z_n be such that $A_\nu, A_{\nu'}$, are isomorphic as S-modules. Then $\nu = \nu'$.

<u>Proof</u>:

Follows by combining the previous <u>two lemmas</u>.

Consider the map

$$Z_n \to \{\text{ irred. reps. of } S \text{ up to isomorphism}\}$$

$$\nu \to A_\nu$$

8)

Theorem. This map is a bijection.

Proof. We already know that this map is a well defined injective map. It is enough to show that it is a map between two finite sets with the same # of elements. We know that
#{irr rep. of S up to iso.} =
#(conj. classes in S)
and this is equal to $\# Z_n$. (Two elements of S are conjugate $\Leftrightarrow$ they have the same cycle type.)

Example (1) If $\nu=(1,1,\dots 1)$ then
$A_\nu=$ constant polynomials = unit rep

(2) if $\nu=(0,0\dots,0,n)$ then
A_ν is spanned by a single element
$\prod_{i<j}(x_i-x_j)$, and it is the sign represent-

(3) If $\nu=(1,1..,1,2)$, $1+1+\dots+1+2=n$, then
A_ν is spanned by x_i-x_j $i\neq j$.

Thus $A_\nu = \{ e_1 x_1 + \dots + e_n x_n \mid e_1 + e_2 + \dots + e_n = 0 \}$ of dimension $n-1$.

4) If $\nu = (0,0,2,2)$, $n=4$, then A_ν is spanned by

$$\underbrace{(x_1 - x_2)(x_3 - x_4)}_{= \alpha}, \quad \underbrace{(x_1 - x_3)(x_2 - x_4)}_{= \beta}, \quad \underbrace{(x_1 - x_4)(x_2 - x_3)}_{= \gamma}$$

with $\alpha - \beta + \gamma = 0$.

This is a 2-dimensional vector space.

<u>Lemma.</u> Let ν, ν' in $\mathbb{Z}_n$ be such that $A_{\nu'} \mid S_\nu$ contains a line on which S_ν acts by $\varepsilon_\nu = \varepsilon \mid S_\nu$ (ε = sign repres. of S.) Then either $\nu = \nu'$ or $|\nu| < |\nu'|$.

10) <u>Proof.</u> By assumption there exists $\Sigma \in A_{\nu'}$, $\Sigma \neq 0$ such that $\rho(s)\Sigma = \varepsilon(s)\Sigma$ for any $s \in S_\nu$. As in the proof of an earlier lemma, this shows that Σ is divisible by φ_ν. Now $\Sigma \in \mathcal{P}_{|\nu'|}$, $\varphi_\nu \in \mathcal{P}_{|\nu|}$. It follows that $|\nu'| \geq |\nu|$.

Assume now that $|\nu'| = |\nu|$.

Then Σ must be a constant multiple of φ_ν so that Σ is a non-zero element of $A_\nu \cap A_{\nu'}$! This implies $\nu = \nu'$ by an earlier lemma. $\square$

<u>Lemma.</u> Let ν be a partition of I. Then $\mathrm{Ind}_{S_\nu}^{S}(\varepsilon_\nu) \cong A_\nu \oplus M$ $(*)$ where M is a direct sum of various $A_{\nu'}$ with $|\nu'| > |\nu|$.

<u>Proof</u> For $\nu' \in \mathcal{Z}_n$ we have

$$(A_{\nu'}; \mathrm{Ind}_{S_\nu}^{S}(\varepsilon_\nu)) = (\varepsilon_\nu; A_{\nu'}|_{S_\nu})$$

and this is zero unless $\nu = \nu'$ or $|\nu'| > |\nu|$ (by previous lemma) and is 1-dimensional if $\nu = \nu'$. $\square$

11)

Cor 1. The elements $\{\mathrm{Ind}_{S_\nu}^{S}(\varepsilon_\nu) \mid \nu \in Z_n\}$ form a $\mathbb{Z}$-basis of the Grothendieck group $R(S)$.

Proof. The previous lemma shows that the collection of these elements is related to the standard basis of $R(S)$ by a triangular matrix with integer entries and with one on diagonal. $\square$

Cor 2. The elements $\{\mathrm{Ind}_{S_\nu}^{S}(1) \mid \nu \in Z_n\}$ where 1 is the unit repres. of S_ν, form a basis of the Grothendieck group $R(S)$.

<u>Proof</u>. If χ is an irred. character of S, then $\chi' : s \to \chi(s)\varepsilon(s)$ is again an irred. char. of S; here $\varepsilon(s) = \mathrm{sign}(s)$. Hence there is a well defined isom.

$\tau : R(S) \to R(S)$ which carries each basis element χ to the basis element χ'. It is enough to show that τ carries

$\chi_{\mathrm{Ind}_{S_\nu}^{S}(\varepsilon_\nu)}$ to $\chi_{\mathrm{Ind}_{S_\nu}^{S}(1)}$. This follows from the equality

$$\mathrm{Ind}_{S_\nu}^{S}(\varepsilon_\nu) \otimes \varepsilon \simeq \mathrm{Ind}_{S_\nu}^{S}\left(\varepsilon_\nu \otimes \varepsilon|_{S_\nu}\right) = \mathrm{Ind}_{S_\nu}^{S}(1).$$

<u>We use $\varepsilon|_{S_\nu} = \varepsilon_\nu$ and $\varepsilon_\nu \otimes \varepsilon_\nu = 1$.</u>

13) **Examples.** $(n=3)$

$\nu=(1,1,1)$, $\lambda=(1,2,3)$

$$A_\nu = \mathrm{Ind}_{S_{111}}^{S}(1) - \mathrm{Ind}_{S_{102}}^{S}(1) - \mathrm{Ind}_{S_{021}}^{S}(1) + \mathrm{Ind}_{S_{003}}^{S}(1)$$

$\nu=(012)$, $\lambda=(0,2,4)$

$$A_\nu = \mathrm{Ind}_{S_{012}}^{S}(1) - \mathrm{Ind}_{S_{003}}^{S}(1)$$

$\nu=(003)$, $\lambda=(0,1,5)$

$$A_\nu = \mathrm{Ind}_{S_{003}}^{S}(1)$$

Other approaches to rep. theory of S.

The oldest result in the theory is due to Frobenius (~ 1880).

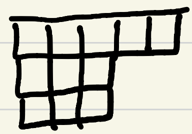
Let $\nu \in \mathbb{Z}_n$. We set

$$\lambda = (\lambda_1, \dots, \lambda_n) = (\nu_1, \nu_2+1, \nu_3+2, \dots, \nu_n+n-1).$$

We have

$$A_\nu = \sum_{\kappa} \mathrm{sign}(\kappa)\, \mathrm{Ind}_{S_{\lambda-\kappa}}^{S}(1) \in R(S)$$

where κ runs over the permutations of $\{0,1,\dots,n-1\}$ such that $\lambda_i - \kappa_i \geq 0$ for all i.

14) A. Young (1927) Gives explicit constr. of irred. rep. of S. Shows that $\dim A_\nu$ = # tableaux 

with rows of lengths:
largest ν_i, next to largest ν_i, ...
in which the numbers $1, 2, .. n$ are inserted so that they increase from left to right and from up to down.
For example if $n=4$:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$$

"standard tableaux"!

<u>Specht</u> ($\sim$ 1932) – these lectures follow essentially his approach.

<u>Springer (1976)</u> Let N be a nilpotent $n \times n$ matrix with Jordan blocks of sizes $a_1 \leq a_2 \leq \dots \leq a_k$.
Consider the set
$\mathcal{B}_N$ of all sequences $V_0 \subset V_1 \subset \dots \subset V_n$ of subspaces of V such that $\dim V_i = i$, $NV_i \subset V_i$ for all i. This is a topological space. Let $H_i(\mathcal{B}_N)$ the homology group of $\mathcal{B}_N$ in degree i with coefficients in $\mathbb{C}$. Then $H_i(\mathcal{B}_N) = 0$ for i odd.

Springer shows that S acts on $H_i(\mathcal{B}_N)$ for any i, and if i is maximum so that $H_i(\mathcal{B}_N)$ then the S·action on $H_i(\mathcal{B}_N)$ is irreducible and each irred. rep. of S is obtained exactly once.