1)

Abelian groups:

If G is a (finite) abelian group then any irred. rep. $\rho : G \to GL(V)$ has dim. 1. Indeed from Schur's lemma $\rho(s)$ is a scalar times 1 for any $s \in G$. Hence any line L in V is invariant so that $L = V$.

Conversely if G is a finite group such that any irred. rep. of G is 1-dimensional then G is abelian.

Indeed, the regular represent. $\rho : G \to GL(V)$ can be written as $V = L_1 \oplus \cdots \oplus L_n$ where L_i are invariant and irred. as repres. of G, hence $\dim L_i = 1$ for s, t in G, $\rho(s), \rho(t)$ act on L_i as scalars hence commute. It follows that $\rho(s)\rho(t) = \rho(t)\rho(s) : V \to V$. Applying this to the basis element $e_1 \in V$ we obtain $e_{st} = e_{ts}$ hence $st = ts$. $\square$

2) <u>Direct product of two groups.</u>

Let G, G' be finite groups. Then $G \times G'$ is a (finite) group with operation $(s,s')(t,t') = (st, s't')$. If $\rho : G \to GL(V)$, $\rho' : G' \to GL(V')$ are repres. of G, G', then $\rho \boxtimes \rho' : G \times G' \to GL(V \otimes V')$
$(s,s') : [v \otimes v' \to \rho(s)v \otimes \rho'(s')v']$ is a lin. rep. of $G \times G'$ with character

(*) $\chi_{\rho \boxtimes \rho'}(s,s') = \chi_\rho(s)\chi_{\rho'}(s')$.

Let $\chi_1, \chi_2, \dots, \chi_n$ be a complete list of the characters of irred. reps. of G.
Let $\chi'_1, \chi'_2, \dots, \chi'_{n'}$ be a complete list of the characters of irred. reps. of G'.

<u>Lemma</u>. A complete list of the characters of irred. reps. of $G \times G'$ is

$\chi_{i,i'} : (s,s') \to \chi_i(s)\chi_{i'}(s')$ $\qquad (1 \le i \le n, 1 \le i' \le n')$.

<u>Proof</u> From (*) we see that $\chi_{i,i'}$ is the char. of a lin. rep. of $G \times G'$. For i, j in $[1,n]$, i', j' in $[1, n']$ we have

$(\chi_{i,i'} | \chi_{j,j'}) = (\chi_i | \chi_j)(\chi_{i'} | \chi_{j'}) = \delta_{ij}\delta_{i'j'}$.

It follows that each $\chi_{ii'}$ is the char. of an irred. rep. of $G \times G'$ and that the various $\chi_{ii'}$ are distinct. To show that they exhaust all irred. reps. it is enough to show

$$\sum_{i,i'} \chi_{i,i'}(1,1)^2 = \#(G \times G')$$

But the left hand side is $\sum_{i,i'} \chi_i(1)^2 \chi_{i'}(1)^2 = \#G \cdot \#G'$. $\square$

<u>Cor.</u> Any irr. rep of $G \times G'$ is of the form $\rho \boxtimes \rho'$ where ρ is an irr. rep. of G and ρ' is an irr. rep. of G'; also ρ, ρ' are well defined up to isomorphism. Conversely if ρ, ρ' are as above, then $\rho \boxtimes \rho'$ is irreducible.

4)

<u>Induced representations</u>. Let H be a subgroup of a finite group G, and let $\rho : H \to GL(V)$ be a lin. repres.

Let $\tilde{V} = \{ f : G \to V ;\ f(gh) = \rho(h)^{-1} f(g) \ \forall g \in G, h \in H \}$

Define $\tilde{\rho} : G \to GL(\tilde{V})$ by

$\tilde{\rho}(g)(f) = f'$, $f \in \tilde{V}$,

$f'(g') = f(g^{-1} g')$

We must check that $f' \in \tilde{V}$ that is

$f(g^{-1} g' h) = \rho(h)^{-1} f(g^{-1} g')$ for $h \in H$.

This is clear. We show that

$\tilde{\rho}(g_1 g_2)(f) = \tilde{\rho}(g_1)\tilde{\rho}(g_2)(f)$ for any $f \in \tilde{V}$, g_1, g_2 in G, that is

$f((g_1 g_2)^{-1} g') = \tilde{\rho}(g_2)(f)(g_1^{-1} g')$

$= f(g_2^{-1} g_1^{-1} g')$. This is clear.

5) Thus, $\tilde\rho$ is a lin. rep. called <u>induced repres</u>: $\tilde\rho = \mathrm{Ind}_H^G(\rho)$.

Let $Z \subset G$ be a set of representatives for the equiv rel. $g \sim g'$ if $g' \in gH$.

Let $\tilde V_0 = \{ f_0 : Z \to V \}$.

The map $\tilde V \xrightarrow{I} \tilde V_0$, $f \to f|_Z$ is an isomorphism of vector spaces. Its inverse $\tilde V_0 \xrightarrow{J} \tilde V$ takes $f_0 \in \tilde V_0$ to $f : G \to V$ where $f(g) = \rho(h)^{-1} f_0(z)$ for $g = zh$. $z \in Z$, $h \in H$. (We have $f(gh') = \rho(hh')^{-1} f_0(z)$ $= \rho(h')^{-1}\rho(h)^{-1} f_0(z) = \rho(h')^{-1} f(g)$ so that $f \in \tilde V$.)

For $g \in G$ we consider the linear map $S : \tilde V_0 \longrightarrow \tilde V_0$, $f_0 \longrightarrow I(\tilde\rho(g)(J(f_0)))$

We have $S(f_0)(z) = (\tilde\rho(g) J(f_0))(z) = J(f_0)(g^{-1}z)$ [write $g^{-1}z = z'h$, $z' \in Z$, $h \in H$] $= \rho(h^{-1}) f_0(z')$.

We have $\tilde V_0 = \bigoplus_{u \in Z} \tilde V_0^u$, $\tilde V_0^u = \{ f_0 \in \tilde V_0 \mid f_0|_{Z - \{u\}} = 0 \}$

6) If $f_0 \in \tilde{V}_0^Z$ then

$$(S f_0)(z) = \begin{cases} \rho(h^{-1}) f_0(z) = \rho(z^{-1} g z) f_0(z) & \text{if } z = z \text{ i.e } z^{-1} g z \in H \\ 0 & \text{otherwise} \end{cases}$$

Hence $\operatorname{tr}(S, \tilde{V}_0) = \sum_{\substack{z \in Z \\ z^{-1} g z \in H}} \operatorname{tr}(\rho(z^{-1} g z) : V \to V)$

$$= \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1} g x \in H}} \operatorname{tr}(\rho(x^{-1} g x) : V \to V).$$

We use : $\operatorname{tr}(h_1^{-1} z^{-1} g z h_1 : V \to V) = \operatorname{tr}(z^{-1} g z : V \to V)$
for $h_1 \in H$

We have $\operatorname{tr}(\tilde{\rho}(g), \tilde{V}) = \operatorname{tr}(S, \tilde{V}_0)$
since I, J are inverse to each other.

<u>Theorem</u>: $\chi_{\tilde{\rho}}(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1} g x \in H}} \chi_\rho(x^{-1} g x)$

for $g \in G$.

7)

Frobenius reciprocity.

Let H be a subgroup of G, let $\rho: H \to GL(V)$, $\rho': G \to GL(V')$ be lin. repres. Then

$$(\mathrm{Ind}_G^H(\rho) ; \rho') = (\rho ; \rho'|_H) \quad \text{or equivalently}$$

$$(\chi_{\mathrm{Ind}_H^G(\rho)} \mid \chi_{\rho'}) = (\chi_\rho \mid \chi_{\rho'|_H}).$$

Here $\rho'|_H : H \to GL(V')$ is the restriction of ρ'.

Proof. We must show:

$$\frac{1}{\# G} \sum_{s \in G} \frac{1}{\# H} \sum_{\substack{t \in G \\ tst^{-1} \in H}} \chi_\rho(tst^{-1}) \chi_{\rho'}(s^{-1})$$

$$= \frac{1}{\# H} \sum_{h \in H} \chi_\rho(h) \chi_{\rho'}(h^{-1}).$$

In the left hand side we change variable

$$\{(s,t) \in G \times G \mid tst^{-1} \in H\} \leftrightarrow \{(h,t) \in H \times G\}$$

$h = tst^{-1}$. The left hand side becomes

$$\frac{1}{\# G \, \# H} \sum_{\substack{t \in G \\ h \in H}} \chi_\rho(h) \chi_{\rho'}(t^{-1} h^{-1} t)$$

and this equals the right hand side since $\chi_{\rho'}(t^{-1}h^{-1}t) = \chi_{\rho'}(h^{-1})$. $\square$

8)

Restriction of an induced representation to a subgroup.

Let H, K be subgroups of G, let $\rho : H \to GL(V)$ be a lin. repres. Let ρ_1 be the restriction of $\mathrm{Ind}_H^G(\rho)$ to K.

Let Ω be a set of representatives for the double cosets $K \backslash G / H$ (equivalence classes for the relation $g \sim g'$ if $g' \in KgH$ on G).

For each $\omega \in \Omega$ let $H_\omega = \omega H \omega^{-1} \cap K$, a subgroup of K. Define a linear rep. of H_ω by $\rho_\omega : H_\omega \to GL(V)$, $\rho_\omega(x) = \rho(\omega^{-1} x \omega)$.

Mackey formula:

ρ_1 is isomorphic to $\displaystyle\bigoplus_{\omega \in \Omega} \mathrm{Ind}_{H_\omega}^K (\rho_\omega)$.

It is enough to verify the equality of characters at $s \in K$:

$$\frac{1}{\# H} \sum_{\substack{t \in G \\ tst^{-1} \in H}} \chi_\rho(tst^{-1}) = \sum_{\omega \in \Omega} \frac{1}{\# H_\omega} \sum_{\substack{k \in K \\ ksk^{-1} \in H_\omega}} \chi_{\rho_\omega}(ksk^{-1})$$

$$\| $$

$$\sum_{\omega \in \Omega} \frac{1}{\# H_\omega} \sum_{\substack{k \in K \\ ksk^{-1} \in \omega H \omega^{-1}}} \chi_\rho(\omega^{-1} k s k^{-1} \omega)$$

9)

In the left hand side we can write $t = h\omega^{-1}k$ for a unique $\omega \in \Omega$ and with $h \in H$, $k \in K$. The left hand side becomes

$$\frac{1}{\# H} \sum_{\substack{h \in H \\ k \in K \\ \omega \in \Omega \\ h\omega^{-1}ksk^{-1}\omega h^{-1} \in H}} \chi_\rho(h\omega^{-1}ksk^{-1}\omega h^{-1}) \frac{1}{\# H_\omega}$$

$$= \sum_{\substack{k \in K \\ \omega \in \Omega \\ \omega^{-1}ksk^{-1}\omega \in H}} \chi_\rho(\omega^{-1}ksk^{-1}\omega) \frac{1}{\# H_\omega}$$

and this is equal to the right hand side. $\square$

10) Let $H, K \subset G$, $\Omega \subset G$ be as above.
Let $\rho: H \to GL(V)$, $\rho': K \to GL(V')$ be lin. repres. By Frobenius and Mackey:

$$\dim\left(\mathrm{Ind}_H^G(\rho) ; \mathrm{Ind}_K^G(\rho')\right) = \dim\left(\mathrm{Ind}_H^G(\rho)|K ; \rho'\right)$$

$$(*) = \sum_{\omega \in \Omega} \dim\left(\mathrm{Ind}_{H_\omega}^K(\rho_\omega) ; \rho'\right) =$$

$$= \sum_{\omega \in \Omega} \dim\left(\rho_\omega ; \rho'_{[\omega]}\right) \qquad (**)$$

where $\rho_\omega : H_\omega \to GL(V)$ is $x \to \rho(\omega^{-1} x \omega)$

$\rho'_{[\omega]} : H_\omega \to GL(V')$ is $x \to \rho'(x)$.

Assume now that $H = K$, $V = V'$, $\rho = \rho'$ are irreducible. The condition that $\mathrm{Ind}_H^G(\rho)$ is irreducible is that $(*)$ equals 1, that is, $(**)$ equals 1. But in $(**)$ we have

$\dim(\rho_w ; \rho_{[w]}) \geq 0$ for $\omega \in \Omega, \omega \notin H$

$= \dim(\rho ; \rho) = 1$ for $\omega \in \Omega, \omega \in H$.

Thus, for $\rho : H \to GL(V)$ irreducible:
$\mathrm{Ind}_H^G(\rho)$ is irreducible $\Leftrightarrow$

for any $s \in G - H$, the representations
$$sHs^{-1} \cap H \to GL(V), \quad h \mapsto \rho(s^{-1}hs)$$
$$sHs^{-1} \cap H \to GL(V), \quad h \mapsto \rho(h)$$
have no common irreducible direct summand.

Let $\rho : H \to GL(V)$, $\tilde{\rho} : G \to GL(\tilde{V})$ be lin. rep. We show:
$$\mathrm{Ind}_H^G(\rho) \otimes \tilde{\rho} \cong \mathrm{Ind}_H^G\left(\rho \otimes \left(\tilde{\rho}|_H\right)\right).$$
Enough: $\forall s \in G$
$$\sum_{\substack{t \in G \\ tst^{-1} \in H}} \chi_\rho(tst^{-1}) \chi_{\tilde{\rho}}(s) = \sum_{\substack{t \in G \\ tst^{-1} \in H}} \chi_\rho(tst^{-1}) \underbrace{\chi_{\tilde{\rho}}(tst^{-1})}_{= \chi_{\tilde{\rho}}(s)}$$

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Let X be a finite set with G-action that is; we are given $G \times X \to X$, $(g, x) \to gx$ with $s(t(x)) = s(t(x))$ for s, t in G, $1x = x$.

Assume that G acts on X transitively that is for $x, y \in X$ $\exists$ $s \in G$ such that $sx = y$. Let $[X]$ be the $\mathbb{C}$-vector space with basis $\{x \mid x \in X\}$. Define a lin rep. $G \to GL[X]$ by $s \to \rho(s) : x \to sx$.

Let $x_0 \in X$ and let $H = \{s \in G;\ s x_0 = x_0\}$.

We have a lin. map

$T : \mathrm{Ind}_H^G (1) \longrightarrow [X]$, $\quad$ (1 = unit rep. of H)

to a function $f : G \to \mathbb{C}$ such that $f(sh) = f(s)$ $\forall$ $s \in G$, $h \in H$, it associates $Tf = \sum_{s \in G} f(s)(s x_0)$. Then $T(gf) = \sum_s f(g^{-1}s)(s x_0) =$

$$= \sum_{s'} f(s')(g s' x_0) = g\, T(f)$$

so that T commutes with the G-actions. It is an isomorphism of representations.

We say that $[X]$ is a permutation representation.

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Let X' be another $\neq \emptyset$ finite set with G-action. The the lin. repres. $\rho' : G \to [X']$ is defined. We have:

$$(*) \quad ([X];[X']) = \#(G\text{-orbits on } X \times X')$$

where G acts on $X \times X'$ by $s : (x, x') \to (sx, sx')$.

Proof. From the definition we have

$$tr(\rho(s)) = \#\{x \in X;\ sx = x\}$$

$$tr(\rho'(s)) = \#\{x' \in X';\ sx = x\}.$$

Hence

$$([X];[X']) = \frac{1}{\#G} \sum_{s \in G} tr(\rho(s)) \overline{tr(\rho'(s))} =$$

$$= \frac{1}{\#G} \sum_{s \in G} \#\{(x, x') \in X \times X' \mid sx = x,\ sx' = x'\}$$

$$= \frac{1}{\#G} \sum_{(x,x') \in G \times G} \#\{s \in G \mid sx = x,\ sx' = x'\} =$$

$$= \frac{1}{\#G} \sum_{(x,x') \in G \times G} \frac{\#G}{\#(\text{orbit of } (x x'))}$$

$$= \#(\text{orbits of } G \text{ on } X \times X') \quad \square.$$

This could be also deduced from the Mackey formula.