1) Characters.

If $A = (a_{ij})$ is an $n \times n$-matrix with $a_{ij} \in \mathbb{C}$ its trace is

$$\operatorname{tr} A = \sum_i a_{ii}.$$

It satisfies $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
(If $A = (a_{ij})$, $B = (b_{ij})$, this is:

$$\sum a_{ij} b_{ji} = \sum b_{k\ell} a_{\ell k} \quad \text{: clear}).$$

If in addition C is invertible then $\operatorname{tr}(CBC^{-1}) = \operatorname{tr}(B)$.
Indeed $\operatorname{tr}(CBC^{-1}) = \operatorname{tr}(C^{-1}CB) = \operatorname{tr}(B)$.

Let V be a vector space of dim. n.
For $T : V \to V$ linear we set $\operatorname{tr}(T) = \operatorname{tr}(A)$ where A is the matrix of T with respect to a basis of V. If we choose a different basis, A is changed into CAC^{-1} where C is an invertible matrix, so $\operatorname{tr}(T)$ is indep. of basis.

2/ Let $\rho: G \to GL(V)$ be a lin. rep. For $s \in G$ let

$$\chi(s) = \chi_\rho(s) = \mathrm{Tr}(\rho(s)).$$

The function $s \to \chi(s)$, $G \to \mathbb{C}$ is called the <u>character</u> of ρ.

<u>Lemma</u> (1) $\chi(1) = \dim V$
(2) $\chi(s^{-1}) = \overline{\chi(s)}$ (complex conj.)
(3) $\chi(tst^{-1}) = \chi(s)$, $\forall s, t$

Proof (1), (3) are immediate. We prove (2).
For a matrix A, $\mathrm{tr}(A)$ is the sum of eigenvalues of A, $\mathrm{tr}(A^{-1})$ is the sum of inverses of eigenvalues of A. Enough to show: If λ is an eigenv. of $\rho(s)$ then $\lambda^{-1} = \bar{\lambda}$. Enough to show: $\lambda^k = 1$ for some $k \geq 1$. Enough to show: $\rho(s)^k = 1$ for some $k \geq 1$. Follows from $s^{\#G} = 1$.

<u>Example</u>: $\chi_{\rho^*}(s) = \chi_\rho(s^{-1})$. (Use that for a lin. map $T: V \to V$, we have $\mathrm{tr}(T^{\mathrm{transp}}) = \mathrm{tr}(T)$.)

3)

If $\rho^1 : G \to GL(V_1)$, $\rho^2 : G \to GL(V_2)$ are lin. rep. then

$$\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}, \quad \chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \chi_{\rho_2}.$$

Follows from definition.

<u>Schur's lemma</u>. Let ρ^1, ρ^2 be as above. Assume they are irreducible. Let $f : V_1 \to V_2$ be a lin. map such that

$$\rho^2(s) f = f \rho^1(s) : V_1 \to V_2 \qquad \forall s \in G.$$

(1) If ρ^1, ρ^2 are not isomorphic then $f = 0$.

(2) If $V_1 = V_2$, $\rho^1 = \rho^2$ then there exists $\lambda \in \mathbb{C}$ such that $f(x) = \lambda x$, $\forall x \in V_1$

<u>Proof</u>. If $f = 0$, result is obvious. Now assume $f \neq 0$. Let $W = \{x \in V_1 \mid f(x) = 0\}$. If $x \in W$, $s \in G$ then $f(\rho^1(s) x) = \rho^2(s)(f(x)) = 0$ so that $\rho^1(s) x \in W$. Thus W is invariant. By irreducibility of V_1 we have $W = 0$ or

4)

$W = V_1$. Since $f \neq 0$ we have $W \neq V_1$ so that $W = 0$ and f is injective.

Let $W' = f(V_1)$. If $x \in V_1$, $s \in G$, we have $\rho^2(s)(f(x)) = f(\rho^1(s)x) \in W'$. Thus $\rho^2(s)W' \subset W'$ and W' is invariant. By irreducibility of V_2 we have $W' = 0$ or $W' = V_2$. Since $f \neq 0$ we have $W' \neq 0$ so that $W' = W_2$ and f is surjective hence an isomorphism. This proves (1).

We prove (2). By linear algebra, there exists $\lambda \in \mathbb{C}$ such that
$U = \{ x \in V_1 \mid f(x) = \lambda x \} \neq 0$.
If $x \in U$, $s \in G$ then $f(\rho^1(s)x) = \rho^1(s) f(x) =$
$= \lambda \rho^1(s) x$ and $\rho^1(s)x \in U$. Thus, U is invariant. By irreducibility, we have $U = 0$ or $U = V_1$. But $U \neq 0$ so that $U = V_1$. $\square$

5)

Cor. Let $h : V_1 \to V_2$ be a lin. map. Let

$$h^0 = \frac{1}{\# G} \sum_{s \in G} \rho^2(s)^{-1} h \rho^1(s) : V_1 \to V_2 .$$

(1) If ρ^1, ρ^2 are not isomorphic then $h^0 = 0$.

(2) If $V_1 = V_2$, $\rho^1 = \rho^2$ then $h x = \lambda x$ for all $x \in V_1$ where $\lambda = \frac{1}{\dim V} \operatorname{tr}(h)$.

Proof. We have $\rho^2(s) h^0 = h^0 \rho^1(s)$:

$$\rho^2(s) h^0 = \frac{1}{\# G} \sum_{t \in G} \rho^2(s t^{-1}) h \rho^1(t) =$$

$$= \frac{1}{\# G} \sum_{u \in G} \rho^2(u^{-1}) h \rho^1(u s) = h^0 \rho^1(s).$$

By Schur's lemma, in (1) we have $h^0 = 0$; in case (2) we have $h^0 = \lambda 1$ for some $\lambda \in \mathbb{C}$. Also

$$\operatorname{tr}(h^0) = \frac{1}{\# G} \sum_{s \in G} \operatorname{tr}\left(\rho^1(s)^{-1} h \rho^1(s)\right) = \operatorname{tr}(h)$$

$$= \lambda \dim V$$

hence (2) holds.

6)

Now pick a basis (e_i^1) for V_1 and a basis (e_j^2) for V_2. For $s \in G$ we have

$$\rho^1(s) e_{i'}^1 = \sum_i a_{i i'}^1(s) e_i^1 , \quad a_{ii'}^1(s) \in \mathbb{C},$$

$$\rho^2(s) e_{j'}^2 = \sum_j a_{j j'}^2(s) e_j^2 , \quad a_{jj'}^2(s) \in \mathbb{C}.$$

Let $h : V_1 \to V_2$ be the lin. map:

$$h(e_u^1) = \sum_v h_{vu} e_v^2 , \quad h_{vu} \in \mathbb{C}.$$

We have

$$h^0(e_u^1) = \frac{1}{\# G} \sum_{\substack{s \in G \\ i}} a_{iu}^1(s) \, \rho^2(s^{-1}) h e_i^1 =$$

$$= \frac{1}{\# G} \sum_{\substack{s \in G \\ v, i}} a_{iu}^1(s) \, \rho^2(s^{-1}) h_{vi} e_v^2 =$$

$$= \frac{1}{\# G} \sum_{\substack{s \in G \\ v, i, j}} a_{iu}^1(s) \, a_{jv}^2(s^{-1}) h_{vi} \, e_j^2$$

In case (1), this is 0 for any h_{vu} hence $\sum_{s \in G} a_{iu}^1(s) a_{jv}^2(s^{-1}) = 0$ $\forall i, j, u, v$.

7)

In case (2), taking $e^1_u = e^2_u$:

$$\frac{1}{\#G} \sum_{\substack{s \in G \\ v,i,j}} a^1_{iu}(s)\, a^1_{jv}(s^{-1})\, h_{vi}\, e^1_j =$$

$$= \frac{1}{n} \sum_r h_{rr}\, e^1_u \quad , (n = \dim V_1)$$

$$\frac{1}{\#G} \sum_{\substack{s \in G \\ v,i}} a^1_{iu}(s)\, a^1_{jv}(s^{-1})\, h_{vi} = \delta_{ju} \sum_r h_{rr}/n$$

$$= \delta_{ju} \sum_{v,i} \delta_{vi}\, h_{vi}/n \qquad \forall j,u, \forall h_{vi}$$

Hence $\frac{1}{\#G} \sum_{s \in G} a^1_{iu}(s)\, a^1_{jv}(s^{-1})$

$$= \delta_{ju}\, \delta_{vi}/n \qquad \forall i,j,v,u.$$

$$= \begin{cases} 1/n & \text{if } j = u, v = i, \\ 0 & \text{otherwise}. \end{cases}$$

8)

We compute $Z = \frac{1}{\#G} \sum_{s \in G} \chi_{\rho^1}(s) \chi_{\rho^2}(s^{-1})$

$= \frac{1}{\#G} \sum_{\substack{s \in G \\ i,j}} a^1_{ii}(s)\, a^2_{jj}(s^{-1})$.

In case (1) this is zero. In case (2) this is $\frac{1}{\#G} \sum_{\substack{s \\ i,i'}} a^1_{ii}(s)\, a^1_{i'i'}(s^{-1}) =$

$= \sum_{i,i'} \delta_{ii'} \delta_{ii'}/n = \sum_i \frac{1}{n} = 1.$

Thus $Z = \begin{cases} 0 & \text{in case (1)} \\ 1 & \text{in case (2).} \end{cases}$

For two functions $f, g : G \to \mathbb{C}$ define $(f|g) = \frac{1}{\#G} \sum_{s \in G} f(s) \overline{g(s)} = \overline{(g|f)}$

bar = complex conjugation

9)

<u>Cor</u> (1) If χ is the character of an irred. rep. of G then $(\chi|\chi) = 1$.

(2) If χ, χ' are the characters of two non isomorphic representations of G then $(\chi|\chi') = 0$

Proof. Use $\overline{\chi(s)} = \chi(s^{-1})$.

(*) Let $\rho: G \to GL(V)$ be a lin. rep, $V = W_1 \oplus \dots \oplus W_n$, W_i irr. rep. of G. Let $\rho': G \to GL(W)$ be an irred. rep. Then $\#\{i \in [1,n],\ W_i \text{ isom. to } W\} = (\chi_\rho | \chi_{\rho'})$.

Proof. Let χ_i be the char. of W_i. Then $(\chi_\rho | \chi_{\rho'}) = \Sigma (\chi_i | \chi_{\rho'})$ and it remains to use the previous Corollary.

<u>Claim</u>. Two lin repres of G with the same character are isomorphic. (Follows from (*))

Let $\chi_1, \dots \chi_h$ be the distinct irred. char. The char of any rep. is of the form $\varphi = m_1\chi_1 + \dots + m_h\chi_h$ and $(\varphi|\varphi) = \Sigma m_i^2$. Hence φ is irred. $\Leftrightarrow$ $(\varphi|\varphi) = 1$.

10)

If $\rho: G \to GL(V)$, $\rho': G \to GL(V')$ are linear representations let

$$(\rho;\rho') = \left\{ f: V \to V' \text{ linear} \,\middle|\, \rho'(s) f = f \rho(s) \ \forall s \in G \right\}$$

We show:

$$\dim (\rho;\rho') = (\chi_\rho \mid \chi_{\rho'}) .$$

<u>Proof</u>. writing ρ, ρ' as direct sums of irred. reprs., we are reduced to the case where ρ and ρ' are irreducible.

If ρ, ρ' are isomorphic, then

$$(\chi_\rho, \chi_{\rho'}) = 1 = \dim (\rho;\rho')$$

(can assume $\rho = \rho'$; use Schur's lemma)

If ρ, ρ' are not isomorphic then

$$(\chi_\rho, \chi_{\rho'}) = 0 = \dim (\rho;\rho) .$$ $\square$

11) The char. of the regular rep is $r_G : G \to \mathbb{C}$
$r_G(1) = \# G$, $r_G(s) = 0$ if $s \neq 1$. We have
$(r_G | \chi_i) = \chi_i(1)$. Hence any irr. rep. is contained in the reg. rep. with multiplicity $\chi_i(1)$.
We have $r_G(s) = \sum_i \chi_i(1)\chi_i(s)$. Hence
$\sum \chi_i(1)^2 = \# G$.

A function $f : G \to \mathbb{C}$ is a <u>class function</u> if $f(tst^{-1}) = f(s)$, $\forall s, t$ in G (equivalently $f(ab) = f(ba)$ $\forall a, b$ in G). Two elements s, s' in G are <u>conjugate</u> if $s' = tst^{-1}$ for some $t \in G$. This is an equivalence relation on G. Equivalence classes called <u>conjugacy classes</u>. A function $f : G \to \mathbb{C}$ is a class function $\Leftrightarrow$ it is constant on each conjugacy class.

For example, the character of a lin. rep. of G is a class function.

Let $f : G \to \mathbb{C}$ be a class function.

For any lin. rep. $\rho : G \to V$ define a lin. map $T_f^\rho : V \to V$ by $T_f^\rho(x) = \sum_{s \in G} \overline{f(s)} \rho(s)(x)$.

For $t \in G$ we have $\rho(t) T_f^\rho = T_f^\rho \rho(t) : V \to V$.

Indeed,

$$T_f^f \rho(t) x = \sum_s \overline{f(s)} \rho(st)(x) = \sum_{s'} \overline{f(t s' t^{-1})} \rho(t s')(x) =$$

$$= \sum \overline{f(s')} \rho(t s')(x) = \rho(t) T_f^\rho(x).$$

If ρ is irred, then $T_f^\rho = \lambda_\rho \cdot 1$, $\lambda_\rho \in \mathbb{C}$ (Schur).

We have $\lambda_\rho \dim V = tr(T_f^\rho) = \sum_{s \in G} \overline{f(s)} \chi_\rho(s)$ (*)

<u>Lemma.</u> Assume that $f : G \to \mathbb{C}$ is a class function such that $(f \mid \chi_\rho) = 0$ for any irred. rep. ρ.

Then (1) $T_f^{\rho'} = 0$ for any lin rep. $\rho' : G \to GL(V)$

(2) $f = 0$.

Proof. Using (*) we see that $T_f^\rho = 0$ if $\rho : G \to GL(V)$ is irreducible.

Taking direct sums, we deduce that $T_f^\rho = 0$ for any lin. rep. ρ of G.
In particular $T_f^\rho : V \to V$ is zero where $\rho : G \to GL(V)$ is the regular repres. of G (with basis $\{e_t \mid t \in G\}$). We have

$$0 = T_f^\rho(e_1) = \sum_{s \in G} \overline{f(s)}\,\rho(s) e_1$$

$$= \sum_{s \in G} \overline{f(s)}\, e_s .$$

Hence $f(s) = 0$ $\forall\, s \in G$.

$\square$.

<u>Theorem</u>. Let $\chi_1, \chi_2, \dots \chi_h$ be the characters of the various irred. repres. of G. They form a basis of H, the vector space of class functions on G.

Proof. Assume $\sum_{i=1}^{h} c_i \chi_i = 0$, $c_i \in \mathbb{C}$. Take $(\ |\chi_j)$.

We get $c_j = 0$, hence $\{\chi_i\}$ are linearly independent. Let H' be the subspace of H spanned by $\{\chi_i\}$. Define a linear map

$$S : H \longrightarrow H'^*, \quad f \longrightarrow [\, \underset{\substack{\cup\!\!\!\!\;| \\ H'}}{h'} \to (f|\overline{h'}) \,].$$

If $f \in \ker(S)$, then $(f|\chi_i) = 0$ for any i so, by an earlier result, $f = 0$. Thus $\ker(S) = 0$ that is S is injective, so $\dim H \leq \dim H'^* = \dim H'$. It follows that $H' = H$. $\square$

Cor. The number of irred. rep. of G (up to isomorphism) is equal to the number of conjugacy classes of G.

15)

The Grothendieck group $R(G)$ can be identified with the subgroup of the vectorspace of class functions $G \to \mathbb{C}$ with $\mathbb{Z}$-basis given by the characters of irreducible representions (= irred. characters).

Now (1) can be viewed as a bilinear pairing $R(G) \times R(G) \to \mathbb{Z}$ such that

$$(\chi|\chi') = \begin{cases} 1 & \text{if } \chi, \chi' \text{ are irred. char.}, \chi = \chi' \\ 0 & \text{if } \chi, \chi' \text{ are irred. char.}, \chi \neq \chi'. \end{cases}$$

<u>Lemma</u>. Let $\mu \in R(G)$ be such that $(\mu|\mu) = 1$. Then $\mu = \pm\chi$ where χ is an irreducible character. If in addition $\mu(1) > 0$ then the sign $\pm$ is $+$.

<u>Proof</u>. Let $\chi_1, \dots, \chi_n$ be the various (distinct) irred. char. of G. We have $\mu = \sum_{i=1}^{n} m_i \chi_i$ where $m_i \in \mathbb{Z}$. From $(\mu|\mu) = 1$ we deduce $\Sigma m_i^2 = 1$. It follows that $m_i = \pm 1$ for $i = i_0$ and $m_i = 0$ for $i \neq i_0$ (some i_0). If $\mu(1) > 0$ then $\pm \chi_{i_0}(1) > 0$. But $\chi_{i_0}(1) > 0$ hence $\pm = +$. $\square$

Isotypic components of a repres. of G

Let $\rho : G \to GL(V)$ be a lin. repres. For any irred. character χ of G we set $V_\chi = \{x \in V \mid T_\chi x = x\}$ where $T_\chi = \frac{\chi(1)}{\# G} \sum_{s \in G} \overline{\chi(s)}\, \rho(s) : V \to V$.

T_χ commutes with any $\rho(t)$, $t \in G$.

$$\sum_s \overline{\chi(s)}\, \rho(s) \rho(t) \overset{?}{=} \sum_s \overline{\chi(s)}\, \rho(t) \rho(s) \Leftrightarrow$$

$$\sum_{s'} \overline{\chi(s' t^{-1})}\, \rho(s') \overset{?}{=} \sum_{s'} \overline{\chi(t^{-1} s')}\, \rho(s') \; ;$$

use $\chi(s' t^{-1}) = \chi(t^{-1} s')$.

If E is an irred. subrep. of V then $T_\chi|_E = 1$ if $\chi = \chi_E$, $T_\chi|_E = 0$ if $\chi \neq \chi_E$ (χ_E: char. of E).

Indeed $T_\chi|_E = \lambda \cdot 1$, $\lambda \in \mathbb{C}$ and

$$\operatorname{tr}(T_\chi|_E) = \frac{\chi(1)}{\# G} \sum_s \overline{\chi(s)}\, \chi_E(s) = \chi(1)\, \delta_{\chi, \chi_E} = \chi(1)\, \lambda$$

Hence $V = \bigoplus_\chi V_\chi$ (χ runs over the irred. char. of G) and V_χ is the sum of all irred. subrep. of V with character χ. The subspaces V_χ are called the isotypic components of V.