1)

## Affine flag manifold

Let $F_q((\varepsilon))$ be the field of power series in $\varepsilon$ with coefficients in $F_q$. A typical element is $\sum_{i\in\mathbb{Z}} c_i \varepsilon^i$, where $c_i \in F_q$ is zero for $i \ll 0$.

Let $V$ be an $F_q((\varepsilon))$-vector space of dimension $n \geq 2$ ($n < \infty$) with a fixed volume form $\omega$ (= basis element of the 1-dim. vector space $\Lambda^n V$, the $n$-th exterior power of $V$). Let $G = GL(V)$.

Let $\mathcal{O} = F_q[[\varepsilon]]$. A lattice $\mathcal{L}$ in $V$ is a free $\mathcal{O}$-submodule of $V$ such that some $\mathcal{O}$-basis of $\mathcal{L}$ is a $F_q((\varepsilon))$-basis of $V$. For a lattice $\mathcal{L}$, the $n$-th exterior power of $\mathcal{L}$ is a lattice in the 1-dim space $\Lambda^n V$ hence is of the form $\mathcal{O}\varepsilon^{-r}\omega$, for a well defined $r \in \mathbb{Z}$; we set $r = \operatorname{vol} \mathcal{L}$. Let $\mathcal{B}$ be the set of all sequences of lattices $(\mathcal{L}_j)_{j\in\mathbb{Z}}$ such that $\mathcal{L}_{j-1} \subset \mathcal{L}_j$, $\operatorname{vol}(\mathcal{L}_j) = j$, $\varepsilon\mathcal{L}_j = \mathcal{L}_{j-n}$ for all $j$. Then $G$ acts transitively on $\mathcal{B}$. (= affine flag manifold).

2) Let $\mathcal{L}_* = (\mathcal{L}_j)_{j\in\mathbb{Z}}$, $\mathcal{L}'_* = (\mathcal{L}'_j)_{j\in\mathbb{Z}}$ be elements of $\mathcal{B}$. For $i, j$ in $\mathbb{Z}$ we set

$$d_{ij} = \dim(\mathcal{L}'_i \cap \mathcal{L}_j / \mathcal{L}'_i \cap \mathcal{L}_{j-1}) \in \{0, 1\}.$$

For $i \in \mathbb{Z}$ let $X_i = \{j \in \mathbb{Z} ; d_{ij} = 1\}$. Then $X_{i-1} \subset X_i$ for all $i$ and $\#(X_i - X_{i-1}) = 1$ for all $i$. Define $a_i \in \mathbb{Z}$ by $X_i = X_{i-1} \amalg \{a_i\}$. Now $X_{i-n} = X_i - n$ hence $a_{i+n} = a_i + n$ for all $i \in \mathbb{Z}$. One can check that $i \to a_i$ is a bijection $\mathbb{Z} \to \mathbb{Z}$. Using the fact that $\mathrm{vol}(\mathcal{L}_j) = \mathrm{vol}(\mathcal{L}'_j) = j$ we see that $\sum_{i=1}^{n} (a_i - i) = 0$. This gives a bijection

$\{$set of $G$-orbits on $\mathcal{B} \times \mathcal{B}\} \leftrightarrow$ affine symmetric group.