Weyl char. formula and its p-analogue.

Let V_β^p be an irreducible rep. of $GL_n(k)$ (k alg. closed of char. $p \geq 0$) such that $\exists\, v \in V_\beta^p$:

$$\underbrace{\begin{pmatrix} x_{11} & & & 0 \\ & x_{22} & & \\ & & \ddots & \\ * & & & x_{nn} \end{pmatrix}}_{b} v = x_{11}^{\beta_1} x_{22}^{\beta_2} \cdots x_{nn}^{\beta_n} v$$

$\forall\, b \in B^-$. Here $\beta : (\beta_1 \leq \beta_2 \leq \dots \leq \beta_n)$, (integers).

If $p = 0$, Weyl character formula is

$$tr\left(\begin{pmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_n \end{pmatrix} : V_\beta^0 \to V_\beta^0\right) = \frac{\begin{vmatrix} x_1^{\beta_1} & x_1^{\beta_2+1} & \cdots & x_1^{\beta_n+n-1} \\ \cdots & \cdots & \cdots & \cdots \\ x_n^{\beta_1} & x_n^{\beta_2+1} & \cdots & x_n^{\beta_n+n-1} \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \cdots & \cdots & & \\ 1 & x_n & \cdots & x_n^{n-1} \end{vmatrix}}$$

(if $x_i \neq x_j$ $\forall\, i \neq j$)

This is a special case of a formula for character of finite dim. reps of a simple Lie algebra (with a purely algebraic proof) and of a char. formula for irred. rep of a compact Lie group. The most elegant proof is based on the Atiyah-Bott-Lefschetz fixed point theorem.

2) In our case the formula can be deduced from the branching formula from $GL_n(K)$ to $GL_{n-1}(K)$.

Let ${}'V^0_{\beta'}$ be the analogous irred. rep. of $GL_{n-1}(K)$, for $\beta' = (\beta'_1 \leq \cdots \leq \beta'_{n-1})$. (Assume $n \geq 2$.) Then

$$V^0_\beta \big|_{GL_{n-1}(K)} \cong \bigoplus_{\beta'} {}'V^0_{\beta'}$$

where β' run over all $\beta'_1 \leq \cdots \leq \beta'_{n-1}$ such that $\beta_1 \leq \beta'_1 \leq \beta_2 \leq \beta'_2 \leq \cdots \leq \beta'_n \leq \beta_n$.
(Gelfand - Zetlin).

It follows that $\dim V^0_\beta$ is the number of diagrams

$$\begin{array}{ccccccc}
\beta_1 & \leq & \beta_2 & \leq \cdots & & \leq & \beta_n \\
 & \rotatebox{90}{$\leq$}\ \beta'_1 \leq & & \rotatebox{90}{$\leq$}\ \beta'_2 \leq & & \rotatebox{90}{$\leq$}\ \beta'_{n-1} \leq & \\
 & & \rotatebox{90}{$\leq$}\ \beta''_1 \leq & & \rotatebox{90}{$\leq$}\ \beta''_2 \cdots \beta''_{n-2} & & \\
 & & & \vdots & & & \\
 & & & \rotatebox{90}{$\leq$}\ \beta^{(n-1)}_1 \leq & & &
\end{array}$$

of integers with fixed $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$.

3)

Weyl's dimension formula

$$(*)\quad \dim V_\beta = \frac{\prod_{1\le i<j\le n}(\beta_j-\beta_i+j-i)}{\prod_{1\le i<j\le n}(j-i)}$$

This can be deduced from the character formula. (Assume $K=\mathbb{C}$) Let $x_i = e^{(i-1)t}$ where $i=1,\dots,n$, $t\in\mathbb{C}-0$. We have $x_i\ne x_j$ for $i\ne j$ and

$$\operatorname{tr}\left(\begin{pmatrix}1 & & & 0\\ & e^t & & \\ & & e^{2t} & \\ 0 & & & \ddots\ e^{(n-1)t}\end{pmatrix}, V^0_\beta\right) = \frac{\begin{vmatrix}1 & 1 & \cdots & 1\\ e^{\beta_1 t} & e^{(\beta_2+1)t} & \cdots & e^{(\beta_n+n-1)t}\\ \cdots & \cdots & & \vdots\\ e^{(n-1)\beta_1 t} & e^{(n-1)(\beta_2+1)t} & \cdots & \vdots\end{vmatrix}}{\begin{vmatrix}1 & 1 & \cdots & 1\\ 1 & e^t & \cdots & e^{(n-1)t}\\ \vdots & & & \cdots\\ 1 & e^{(n-1)t} & \cdots & e^{(n-1)^2 t}\end{vmatrix}}$$

$$= \frac{\prod_{i<j}\left(e^{(\beta_j+j-1)t} - e^{(\beta_i+i-1)t}\right)}{\prod_{i<j}\left(e^{(j-1)t}-e^{(i-1)t}\right)} = \frac{\prod_{i<j}(\beta_j-\beta_i+j-i)\,t + ct^2+\dots}{\prod_{i<j}(j-i)\,t + c't^2+\dots}$$

Letting $t\to 0$ we get $(*)$.

4) Let $E = \mathbb{R}^n / \{(c, c, \dots, c) \,;\, c \in R\}$ an $n-1$ dimensional R-vector space. Let

$$R^n_{dom} = \{(\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{Z}^n \,;\, \beta_1 \leq \beta_2 \leq \dots \leq \beta_n\}$$

and let E_{dom} be the set of equivalence classes on R^n_{dom} under $(\beta_1, \beta_2, \dots, \beta_n) \sim (\beta_1 + c, \beta_2 + c, \dots, \beta_n + c)$ for $c \in \mathbb{Z}$. The inclusion $R^n_{dom} \subset R^n$ induces an inclusion $E_{dom} \subset E$. Let $\pi : R^n_{dom} \to E_{dom}$ be the ~~ob~~ obvious map. For $\beta \in R^n_{dom}$ we denote $[\beta] = \pi(\beta)$.

For any rational repres. E of $GL_n(K)$ there is a well defined element

$$\chi_E \in \mathbb{N}[X_1, X_2, \dots X_{n-1}, X_1^{-1}, X_2^{-1}, \dots X_{n-1}^{-1}]$$

such that for any $(x_1, \dots x_{n-1}) \in (K^*)^{n-1}$, the trace of

$$\begin{pmatrix} x_1 & & & 0 \\ & \ddots & & \\ & & x_{n-1} & \\ 0 & & & (x_1 x_2 \dots x_{n-1})^{-1} \end{pmatrix}$$

on E is χ_E with X_i replaced by x_i $(1 \leq i \leq n-1)$.

5) In particular $\chi_{V^p_\beta}$ is defined for any $\beta \in R^n_{dom}$, $p = 0$ or a prime number. Note that $\chi_{V^p_\beta}$ depends only on $[\beta]$. Indeed if $[\beta] = [\beta']$ then $V^p_{\beta'} = V^p_\beta$ tensored with a one dimensional representation $GL_n(k) \to k^*$, $g \to \det(g)^m$ under which
$$\begin{pmatrix} x_1 & & & 0 \\ & \ddots & & \\ & & x_{n-1} & \\ 0 & & & (x_1 \cdots x_{n-1})^{-1} \end{pmatrix}$$
is mapped to 1. Thus we write $\chi(p, [\beta])$ instead of $\chi_{V^p_\beta}$.

The affine symmetric group.

We now assume that $\xi \in \mathbb{R}_{\geq 0}$ is fixed.

6)

Let Γ_ξ be the group consisting of all bijection $E \to E$ of the form

$$\tau_\sigma^c \cdot (e_1, e_2, \dots, e_n) \to (e_{\sigma(1)} + c_1\xi, e_{\sigma(2)} + c_2\xi, \dots, e_{\sigma(n)} + c_n\xi)$$

where σ is a permutation of $\{1, 2, \dots, n\}$ and $c = (c_1, c_2, \dots, c_n) \in Z^n$ satisfy $\sum c_i = 0$.

The composition $\tau_\sigma^c \; \tau_{\sigma'}^{c'}$ is

$$(e_1, e_2, \dots, e_n) \to (e_{\sigma\sigma'(1)} + c_{\sigma'(1)}\xi + c_1'\xi, \; {}_{\sigma\sigma'(2)} + c_{\sigma'(2)}\xi + c_2'\xi, \dots)$$

hence it equals

$$\tau_{\sigma\sigma'}^{c''} \qquad \text{where } c'' = (c_{\sigma'(1)} + c_1', c_{\sigma'(2)} + c_2', \dots).$$

Thus, if $\xi \neq 0$, Γ_ξ is the semidirect product of the symmetric group S in n letters with a normal abelian group isomorphic to Z^{n-1}; if $\xi = 0$, we have $\Gamma_\xi = S$.

Recall that the affine symmetric group in n letters, S_{af} is the group of all permutations $\mu : Z \to Z$ such that $\mu(z+n) = \mu(z) + n$ for all $z \in Z$ and $\sum_{z=1}^{n} (\mu(z) - z) = 0$.

This contains S as the subgroup of all $\mu : Z \to Z$ of the form $i \to \sigma(i)$ $(1 \leq i \leq n)$, $i + kn \to \sigma(i) + kn$ $(k \in Z)$ where $\sigma \in S$. This also contains the normal subgroup consisting of all $\mu : Z \to Z$ of the form

7) $\varepsilon_c : i \to i + n c_i$ where $c_i \in \mathbb{Z}$, $\Sigma c_i = 0$.
Then S_{af} is the semidirect product of S with this normal subgroup. We have
$\varepsilon_c \sigma : i \to \sigma(i) + n c_i$. Clearly if $\xi > 0$ there is a well defined group isomorphism
$S_{af} \xrightarrow{\sim} \Gamma_\xi$ given by $\varepsilon_c \sigma \to \tau^c_\sigma$. Hence Γ_ξ is a Coxeter group on the generators $\tau^0_{12}, \tau^0_{23}, \dots \tau^0_{n-1 n}, \tau^1_{1n}$
(where ij denotes the transposition $i \leftrightarrow j$)
with fixed point sets $H^0_{12}, H^0_{23}, \dots, H^0_{n-1n}, H^{-1}_{1,n}$ respectively where

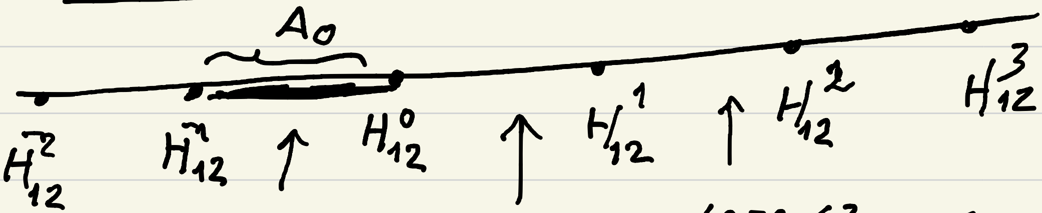
$$H^k_{ij} = \frac{\{(e_1, e_2, \dots, e_n) \in \mathbb{R}^n \mid e_j - e_i = k\xi\}}{\{(c, c, \dots c) \mid c \in \mathbb{R}\}}$$

is a hyperplane in E (not necessarily containing 0.) Here $i<j$ in $\{1, \dots, n\}$ and $k \in \mathbb{Z}$.
The set $E - \bigcup_{\substack{i<j \\ k}} H^k_{ij}$ is a union of connected subsets called <u>alcoves</u>. For example

$$A_0 = \{(e_1, \dots, e_n) \in \mathbb{R}^n, \ e_1 - e_2 < 0, \ e_2 - e_3 < 0, \dots \ e_{n-1} - e_n < 0, \ e_1 - e_n > -1\}$$

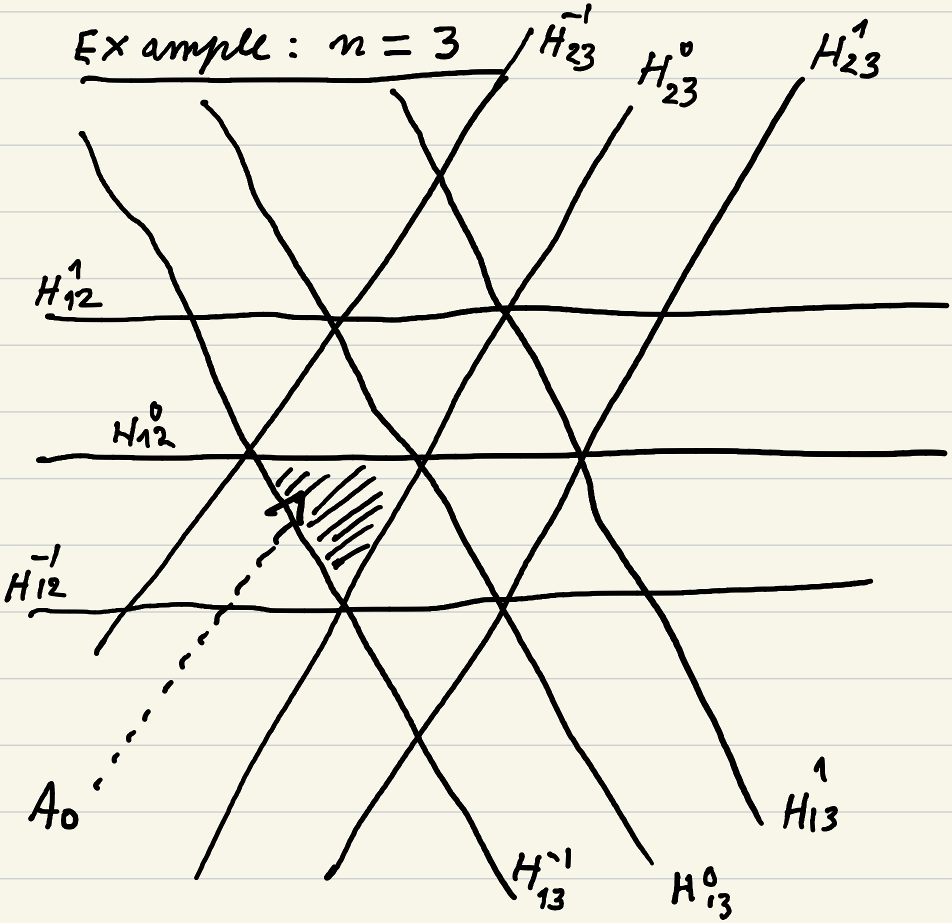
is an alcove.

Example: $n=2$



$\ldots, -1 < e_1 - e_2 < 0, \; 0 < e_1 - e_2 < 1, \; 1 < e_1 - e_2 < 2, \ldots$

Example: $n=3$



9) If $\tau \in \Gamma_\xi$ then τ applied to any alcove is an alcove. This defines an action of $S_{af} = \Gamma_\xi$ on the set of alcoves. This action defines a bijection

$$S_{af} = \Gamma_\xi \longrightarrow \text{set of alcoves}$$

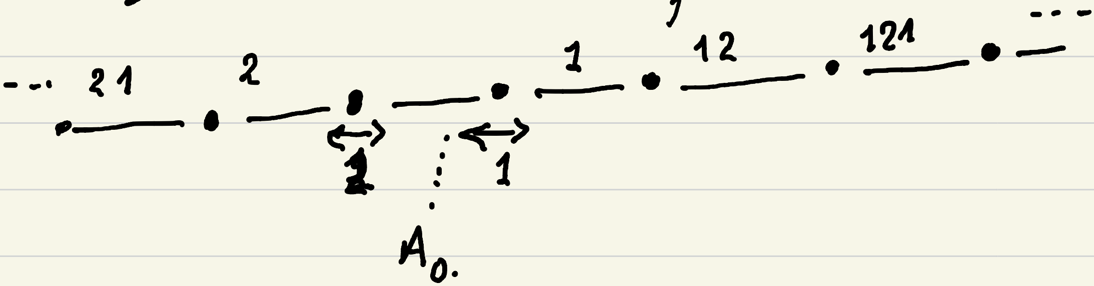
$$\tau \longrightarrow \tau(A_0).$$

We denote the generators

$\tau^0_{12}, \tau^0_{23}, \dots, \tau^0_{n-1,n}, \tau^1_{1n}$ by

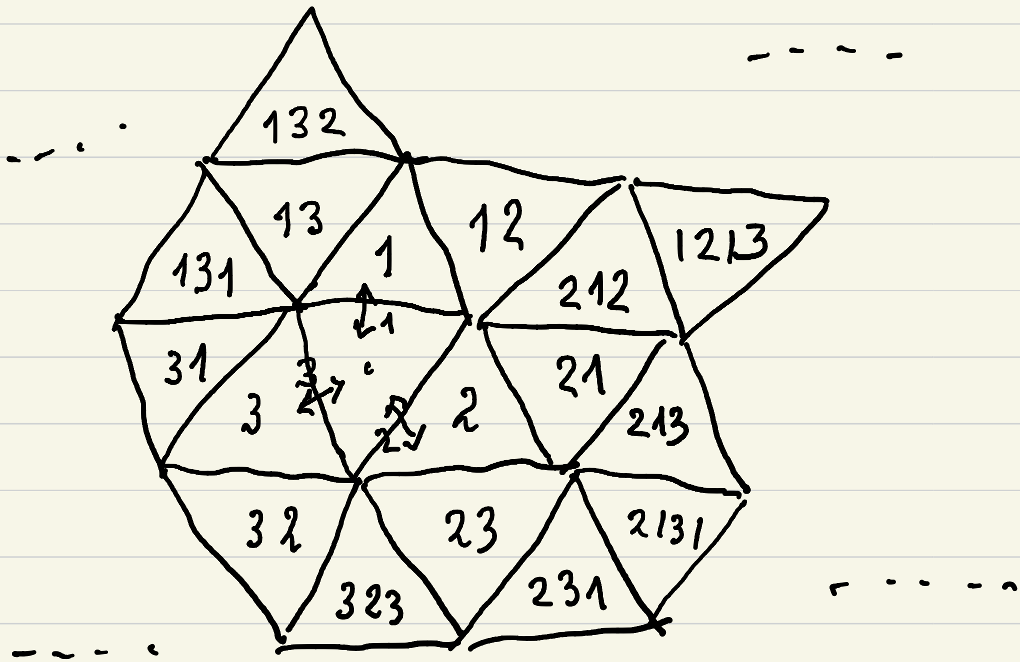
$1 \quad 2 \quad \dots \quad n-1 \quad , n.$

When $n=2$ the correspondence $\Gamma_\xi \Leftrightarrow$ alcoves is as follows:



10)

When $n=3$, the correspondence $\Gamma_\xi \leftrightarrow$ alcoves is as follows



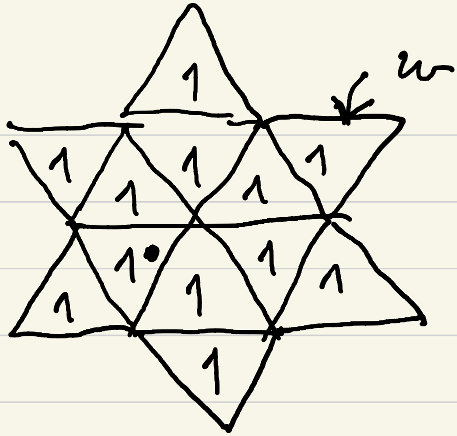
Recall the new basis $\{c_w\}$ of the Hecke algebra corresponding to the affine symmetric group is of the form

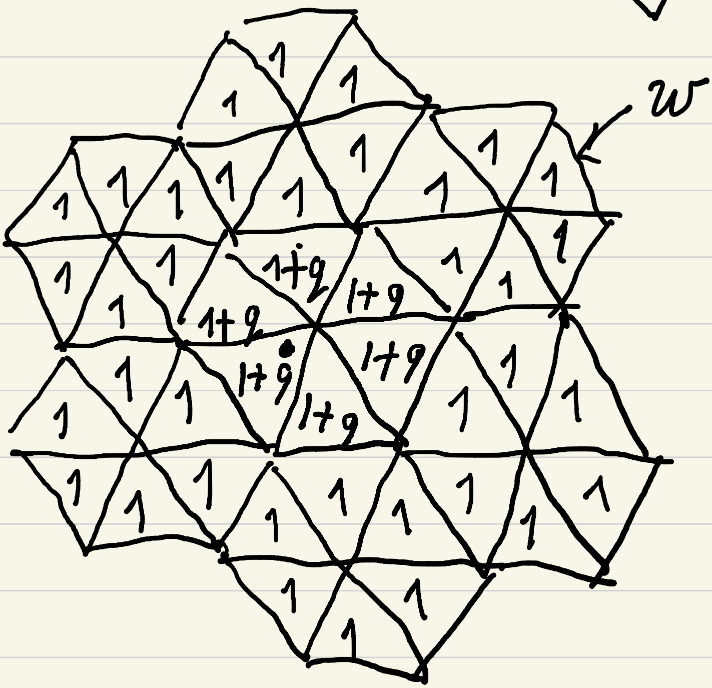
$$c_w = \sum_{y \leq w} v^{-|w|+|y|} P_{yw}(q) T_y$$

where P_{yw} are polynomials in $q = v^2$.

11) Examples

$(n=3)$





alcove corresp. to $y=1$ has • inside

In the alcove corresponding to y the value P_{yw} is written. The alcoves which don't appear have $P_{yw}=0$.

12) Recall that for any $e \in E_{dom} \subset E$ and any $p \in \{0, \text{prime numbers}\}$ we have well defined elements

$\chi(p,e) \in \mathbb{N}[x_1, \dots x_{n-1}, x_1^{-1}, \dots x_{n-1}^{-1}]$

which describe the irred. rational repres. of $GL_n(K)$ with K of char p. These elements are known explicitly when $p=0$ (Weyl char. formula) but not when $p \neq 0$. We denote by $\chi(p,e)_1$ the value of $\chi(p,e)$ at $x_1 = \dots = x_{n-1} = 1$. (This is the dimension of a rational repres.).

Assume now that p is a prime number. Let $t = (1, 2, 3, \dots n) \in E$. We set

$$'H_{ij}^k = H_{ij}^k - t = \frac{\{(e_1, e_2, \dots, e_n) \in E;\ e_j - e_i + j - i \in kp\}}{\{(c, c, \dots c) \mid c \in R\}}$$

for $i<j$, $k \in \mathbb{Z}$. The connected components of $E - \bigcup_{\substack{i<j \\ k}} {}'H_{ij}^k$ are $'A_y = A_y - t$ $(y \in S_{aff} = \Gamma_p)$

where A_y is the alcove indexed by $y \in S_y$.

13)

Let $e \in E_{dom}$. Assume that $e \in {}'A_w$, $w \in \Gamma_p$. For any $y \in \Gamma_p$ there is a unique element $e_y \in {}'A_y$ such that $e_y + t$, $e + t$ are in the same Γ_p-orbit in E.

Assume also that $p \gg 0$ and

(*) $e_1 - e_n \lesssim p^2 - \text{const} \cdot p$.

<u>Theorem</u> $$\chi(p,e) = \sum_{\substack{y \leq w \\ e_y \in E_{dom}}} \operatorname{sgn}(yw) P_{y,w}(1)\, \chi(0, e_y)$$

Conjectured in 1979

Proved in 1994-95:

Andersen - Jantzen - Soergel

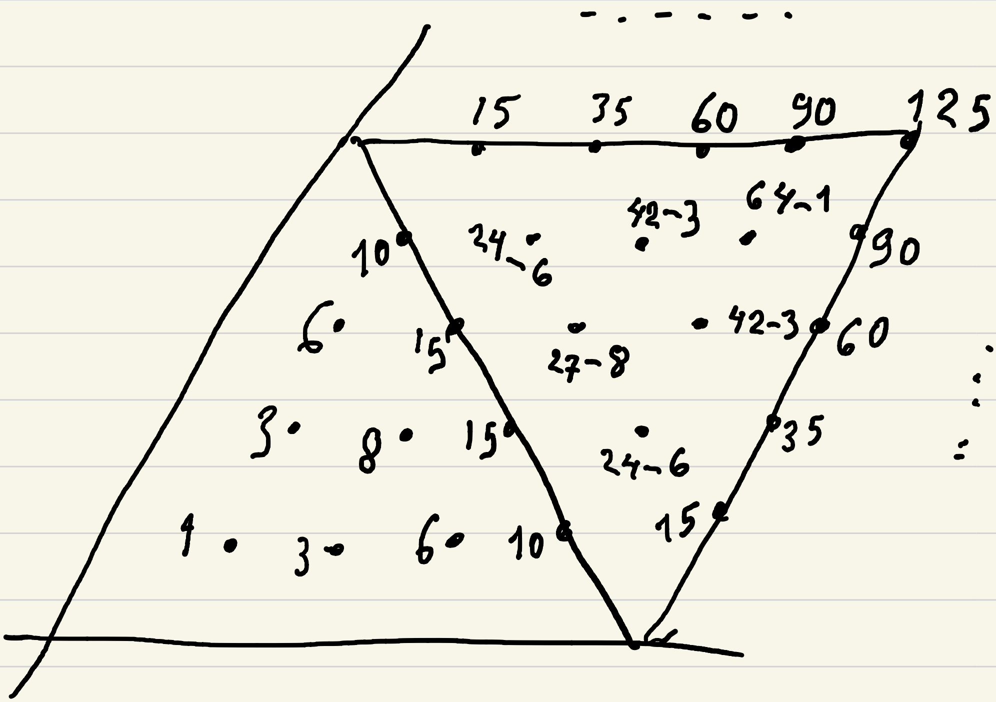
Kazhdan - Lusztig

Kashiwara - Tanisaki

It is possible to reduce the calculation of $\chi(p,e)$ in the case where e is not in an alcove to the case where it is. One can reduce the general case to the case where (*) holds. But the assumption $p \gg 0$ cannot be removed.

14)

Dimension formulas, $\chi(p,e)_1$ for $n=3$, $p=5$



$$\chi(0,e) = \frac{ab(a+b)}{2}, \quad a \geq 1,\ b \geq 1$$

Dimension formulas, $\chi(p,e)$ for $n=2$, $p=3$