1)

Symplectic group.

Let V be a vector space of dim N over a field K. A bilinear form $\langle , \rangle : V \times V \to K$. is said to be symplectic if $\langle x, x\rangle = 0 \ \forall x \in V$. (This implies $\langle x, y\rangle = -\langle y, x\rangle$ for $x, y \in V$.) We say that $\langle , \rangle$ is non-degenerate if $\langle x, y\rangle = 0$ $\forall y \Rightarrow x = 0$; $\langle x, y\rangle = 0, \forall x \Rightarrow y = 0$. Assume that on V we are given a fixed non-degenerate symplectic form $\langle , \rangle$. Then $N = 2n$. Let

$Sp(V) = \{g \in GL(V) \mid \langle gx, gy\rangle = \langle x, y\rangle, \forall x, y \text{ in } V\}$,

the symplectic group.

For a subspace V' of V we set

$V'^{\perp} = \{x \in V \mid \langle x, y\rangle = 0, \forall y \in V'\}$. Then

$\dim V' + \dim V'^{\perp} = 2n$. A complete flag

$V_0 \subset V_1 \subset \ldots \subset V_{2n}$ in V is said to be *isotropic* if

$V_i^{\perp} = V_{2n-i}$ for $i = 0, \ldots, 2n$. (It follows that

$V_i \subset V_i^{\perp}$ for $i = 0, \ldots, n$ so that $\langle , \rangle : V_i \times V_i \to K$ is 0.)

Let $\mathcal{B}_{is}$ be the set of isotropic flags in V.

Now $Sp(V)$ acts on $\mathcal{B}_{is}$ by

$g : (V_0 \subset V_1 \subset \ldots \subset V_{2n}) \mapsto (gV_0 \subset gV_1 \subset \ldots \subset gV_{2n})$.

2)

This action is transitive. We describe the $Sp(V)$-orbits on $\mathcal{B}_{is} \times \mathcal{B}_{is}$. For $V_* = (V_0 \subset \dots \subset V_{2n})$, $V'_* = (V'_0 \subset \dots \subset V'_{2n})$ in $\mathcal{B}_{is}$, the permutation σ of $\{1, 2, \dots, 2n\}$ is defined by

$$V_{\sigma(i)-1} \cap V'_i < V_{\sigma(i)} \cap V'_i \quad \text{for } i = 1, \dots, 2n.$$

We claim that σ satisfies $\sigma(2n+1-i) = 2n+1-\sigma(i)$ for $i = 1, \dots, 2n$. In other words we have

$$V_{2n+1-\sigma(i)-1} \cap V'_{2n+1-i} < V_{2n+1-\sigma(i)} \cap V'_{2n+1-i}$$

for $i = 1, \dots, 2n$, that is

$$V_{\sigma(i)}^{\perp} \cap V'^{\perp}_{i-1} > V^{\perp}_{\sigma(i)-1} \cap V'^{\perp}_{i-1},$$

that is

$$(*) \quad V_{\sigma(i)} + V'_{i-1} > V_{\sigma(i)-1} + V'_{i-1}.$$

Recall that there are lines $L_1, \dots, L_{2n}$ such that $V_i = L_1 \oplus \dots \oplus L_i$, $V'_i = L_{\sigma(1)} \oplus \dots \oplus L_{\sigma(i)}$.

Then $V_{\sigma(i)} + V'_{i-1} = (L_1 + \dots + L_{\sigma(i)}) + (L_{\sigma(1)} + \dots + L_{\sigma(i-1)})$ contains $L_{\sigma(i)}$ and

$V_{\sigma(i)-1} + V'_{i-1} = (L_1 + \dots + L_{\sigma(i)-1}) + (L_{\sigma(1)} + \dots + L_{\sigma(i-1)})$ does not contain $L_{\sigma(i)}$. Thus $(*)$ holds.

3)

We see that $\mathrm{pos}(V_*, V'_*) \in S$ for two isotropic flags belongs to the subgroup $S_{is} = \{\sigma \in S \mid \sigma(2n+1-i) = 2n+1-\sigma(i), \forall i\}$ of S. For any $\sigma \in S_{is}$ we denote by $\mathcal{O}_{is,\sigma}$ the set of all $(V_*, V'_*) \in \mathcal{B}_{is} \times \mathcal{B}_{is}$ such that $\mathrm{pos}(V_*, V'_*) = \sigma$. One can show that $\mathcal{O}_{is,\sigma}$ $(\sigma \in S_{is})$ are precisely the orbits of $Sp(V)$ on $\mathcal{B}_{is} \times \mathcal{B}_{is}$.