1)

Rational repres. of $GL_2(K)$

Here K is an alg. closed field of char. $p \geq 0$. For $l \geq 0$ let P_l be the vector space of homogeneous polynomials of degree l in two variables X_1, X_2, i.e. expressions

$$\pi = \sum_{0 \leq i \leq l} a_i X_1^i X_2^{l-i}, \quad a_i \in K.$$

Now $GL_2(K)$ acts on P_l as follows

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \pi = \sum_{0 \leq i \leq l} a_i (x_{11} X_1 + x_{21} X_2)^i (x_{12} X_1 + x_{22} X_2)^{l-i}.$$

This is a rational repres. of $GL_2(K)$.

Let M_l be the subspace of P_l consisting of all polynomials of the form

$$\sum_{\alpha, \beta} (\alpha X_1 + \beta X_2)^l.$$

This is the smallest subset of P_l invariant under $GL_2(K)$ and containing X_1^l. It is a rational repres. of $GL_2(K)$. For $u \in K$, $f \in M_l$ we have $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} f = \sum_{n \geq 0} u^n A_n f$ where

2)

$A_n : M_\ell \to M_\ell$ are linear maps. In particular

$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} X_2^\ell = (uX_1 + X_2)^\ell = \sum_{0 \leq n \leq \ell} u^n X_1^n X_2^{\ell-n} \binom{\ell}{n}$

$= \sum u^n A_n(X_2^\ell)$. Hence $\binom{\ell}{n} X_1^n X_2^{\ell-n} \in M_\ell$ for any $n = 0, 1, \dots, \ell$. Conversely we have

$(\alpha X_1 + \beta X_2)^\ell = \sum \alpha^n \beta^{\ell-n} \binom{\ell}{n} X_1^n X_2^{\ell-n}$ hence the elements $\binom{\ell}{n} X_1^n X_2^{\ell-n}$ generate M_ℓ. Thus M_ℓ has basis the elements $X_1^n X_2^{\ell-n}$ such that $0 \leq n \leq \ell$ and $\binom{\ell}{n} \neq 0$ in K.

If $p=0$ we have always $\binom{\ell}{n} \neq 0$ for $0 \leq n \leq \ell$ hence $M_\ell = P_\ell$.

Suppose now that $p>0$. We write

$\ell = \sum_{i \geq 0} \ell_i p^i$, $0 \leq \ell_i \leq p-1$, $n = \sum_{i \geq 0} n_i p^i$,

$0 \leq n_i \leq p-1$. Then $\binom{\ell}{n} = \binom{\ell_0}{n_0}\binom{\ell_1}{n_1} \cdots$ and this is $\neq 0 \iff n_0 \leq \ell_0, n_1 \leq \ell_1, n_2 \leq \ell_2, \dots$ It follows that $\dim M_\ell = (\ell_0 + 1)(\ell_1 + 1)\cdots$

In order that $M_\ell = P_\ell$ we need that

$n \leq \ell \Rightarrow n_0 \leq \ell_0, n_1 \leq \ell_1, n_2 \leq \ell_2, \dots$ (*)

3)

This holds if

$$l = (p-1) + (p-1)p + \dots + (p-1)p^{k-1} + l_k p^k$$
$$= (l_k+1)p^k - 1 \qquad \text{with} \qquad 0 \leq l_K \leq p-1$$

<u>Lemma.</u> P_l has a $\underbrace{\text{unique}}^{\neq 0}$ element invariant under all $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ (up to scalar).

<u>Proof</u> We have clearly $\begin{pmatrix} 1 & \\ u & 1 \end{pmatrix} X_2^l = X_2^l$. and

$$\begin{pmatrix} 1 & \\ u & 1 \end{pmatrix} \underbrace{\sum \alpha_i X_1^i X_2^{l-i}}_{\pi} = \sum \alpha_i (X_1 + uX_2)^i X_2^{l-i} =$$
$$= \sum_{0 \leq j \leq i \leq l} \alpha_i X_1^j X_2^{l-j} u^{i-j} \binom{i}{j}$$

Hence π is invariant $\Leftrightarrow \displaystyle\sum_{j \leq i \leq l} \alpha_i \binom{i}{j} u^{i-j} = \alpha_j$ for $j = 0, \dots, n$

$\Leftrightarrow \alpha_i \binom{i}{j} = 0$ for $i > j$ $\Leftrightarrow \alpha_i = 0$ for $i > 0$

$\Leftrightarrow \pi = \alpha_0 X_2^l$. $\square$

<u>Propos.</u> M_l is irreducible for $l \geq 0$.

Proof. Assume $M' \subset M$ is a $\neq 0$ subrepres. We can find $v \neq 0$ in M' invariant under all $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$. By the lemma, $v = \alpha_0 X_2^l$ $\alpha_0 \in K^*$. It follows that $M' = M$. $\square$