1)

Rational repres. of $GL_n(K)$ (after Chevalley)

Let K be an algebraically closed field of characteristic $p \geq 0$. Let $f: GL_n(K) \to K$ be a function. We say that f is *regular* if f is a polynomial in the entries x_{ij} and in $\|x\|^{-1}$ where $x = \det(x_{ij})$. A homomorphism $GL_n(K) \xrightarrow{\rho} GL_m(K)$ is said to be *rational* if for any i,j in $1,\dots,m$ the function $g \to \rho(g)_{ij}$ is in $Reg(GL_n(K))$, the space of regular fns. on $GL_n(K)$.

Lemma The rational homom. $\varphi: K^* \to K^*$ are precisely the maps $x \to x^n$, $n \in \mathbb{Z}$.

Proof Let $\varphi: K^* \to K^*$ be a rat. hom. We have $\varphi(x) = \sum_i \alpha_i x^i$, $\alpha_i \in K$ (finite sum). We have $\varphi(1) = 1$ hence $\sum \alpha_i = 1$. We have $\varphi(xy) = \varphi(x)\varphi(y)$, $\sum \alpha_i x^i y^i = \sum \alpha_i \alpha_j x^i y^j$ hence $\alpha_i^2 = \alpha_i$, $\alpha_i \alpha_j = 0$ for $i \neq j$. Thus $\alpha_i \in \{0,1\}$ and not more than one α_i is 1. Since $\sum \alpha_i = 1$, we see that exactly one α_i is 1. □

Lemma The rational homom. $\varphi: K^* \to GL_m(K)$ are precisely the maps $t \to \sum_i t^i \varphi_i$ (finite sum) where $\varphi_i \in Mat_{m \times m}(K)$ satisfy

2)

$\sum_i \varphi_i = 1$, $\varphi_i \varphi_j = 0$ for $i \neq j$, $\varphi_i^2 = \varphi_i$

Remark. Set $V_i := \{x \in K^m \mid \varphi_i(x) = x\}$. Then $K^m = \bigoplus_i V_i$ and $\varphi_i \mid V_i = t^i$.

<u>Lemma</u> Let $\varphi : K \to GL_m(K)$ be a homom. of the form $t \to \varphi(t) = \sum_{i \geq 0} \varphi_i t^i$ (finite sum) where each φ_i is in $M_{m \times m}(K)$.

If $p = 0$ then $\varphi(t) = e^{tA}$ for some $m \times m$ matrix A with $A^m = 0$

If $p > 0$ then $\varphi(t) = e^{tA_1} e^{t^p A_p} e^{t^{p^2} A_{p^2}} \cdots$ where $A_{p^i}^p = 0$, $A_{p^i} A_{p^j} = A_{p^j} A_{p^i}$

<u>Proof</u> Since $\varphi(0) = 1$, we have $\varphi_0 = 1$. We have $\varphi(t+s) = \varphi(t)\varphi(s)$, $\sum \varphi_i (t+s)^i = \sum \varphi_i t^i \sum \varphi_j t^j$ hence $\varphi_i \varphi_j = \frac{(i+j)!}{i!\,j!} \varphi_{i+j}$. This proves the result when $p = 0$; we have $\varphi_1 = A$, $\varphi_n = A^n/n!$

Assume now $p > 0$. Let $\varphi_{p^i} = A_{p^i}$. Clearly $A_{p^i} A_{p^j} = A_{p^j} A_{p^i}$, $(\varphi_{p^i})^2 = \binom{2p^i}{p^i} \varphi_{2p^i} = \binom{2}{1} \varphi_{2p^i} = 2\varphi_{2p^i}$, $(\varphi_{p^i})^3 = 2\varphi_{p^i}\varphi_{2p^i} = 2\binom{3p^i}{p^i}\varphi_{3p^i} = 2 \cdot 3 \varphi_{3p^i}$ and in general $\varphi_{\alpha p^i} = \frac{1}{\alpha!}(\varphi_{p^i})^\alpha$ for $\alpha = 0, 1, \ldots, p-1$.

3)

Also $\varphi_{\alpha_0+\alpha_1 p+\alpha_2 p^2+\dots} = \varphi_{\alpha_0}\varphi_{\alpha_1 p}\cdots\varphi_{\alpha_i p^i}\cdots$ for α_i in $\{0,1,\dots,p-1\}$. This follows from

$$\binom{\alpha_0+\alpha_1 p+\alpha_2 p^2+\dots}{\beta_0+\beta_1 p+\beta_2 p^2+\dots} = \binom{\alpha_0}{\beta_0}\binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\cdots \bmod p$$

for $0\leq \alpha_i \leq p-1$, which follows from

$$(1+x)^{\alpha_0+\alpha_1 p+\alpha_2 p^2+\dots} = (1+x)^{\alpha_0}(1+x^p)^{\alpha_1}(1+x^{p^2})^{\alpha_2}\cdots$$

clearly $(\varphi_{p^i})^p = p!\,\varphi_{p^{i+1}} = 0$. $\square$

<u>Lemma.</u> Let $\varphi: K\to GL_m(K)$ be as in previous lemma. Assume $m>0$. Then there exists $x\in K^m-\{0\}$ such that $\varphi(t)x=x$ for all $t\in K$

<u>Proof</u> When $p=0$ we have $\varphi(t)=e^{tA}$ where A is nilpotent $m\times m$-matrix. Can find $x\in K^m-\{0\}$ with $Ax=0$. Then $\varphi(t)x=x$, $\forall t$.

Assume $p>0$. Since $A_1^p=0$, the subspace $\{x\in K^m, A_1x=0\}$ is $\neq 0$. It is stable under A_p, with $A_p^p=0$. Hence the subspace $\{x\in K^m / A_1x=A_px=0\}$ is $\neq 0$. Continuing we see that $\{x\in K^m / A_1x=A_px=A_{p^2}x=\dots=0\}$ is $\neq 0$. If $x\neq 0$ is in this subspace then $\varphi(t)x=x$, $\forall t$. $\square$

4)

Let $U = \left\{ (x_{ij}) \in GL_n(K) ; \begin{array}{l} x_{ij} = 0 \text{ for } i > j \\ x_{ii} = 1 \ \forall i \end{array} \right\}$. Let $\rho : U \to GL_m(K)$

be a homomorphism such that for any k, l in $1, \dots, m$, $u \mapsto \rho(u)_{k,l}$ is a polynomial in x_{ij}, $i<j$.

<u>Lemma.</u> Assume that $m > 0$. There exists $x \in K^n \setminus \{0\}$ such that $\rho(u)x = x$, $\forall\, u \in U$.

<u>Proof</u> Consider the sequence $\{1\} = G_0 < G_1 < G_2 < \dots < G_N = U$ of subgroups of U where

$$G_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & * \\ & 1 & & & 0 \\ & & 1 & & \vdots \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & 0 & \cdots & * & * \\ & 1 & & & 0 \\ & & \ddots & & 0 \\ & & & & 1 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 1 & 0 & & * & * \\ & \ddots & \ddots & 0 & * \\ & & & & \\ & & & & 1 \end{pmatrix},$$

$$G_4 = \begin{pmatrix} 1 & 0 & & 0 & * & * & * \\ & \ddots & & & 0 & 0 & * \\ & & & & & 0 & 0 \\ & & 1 & & & & \\ & & & & \ddots & & 1 \end{pmatrix}, \dots$$ Then G_i is normal in G_{i+1}

and $G_{i+1}/G_i = K$. By a previous lemma, the space $V_1 = \{ v \in K^m ; \rho(t)v = v, \forall\, t \in G_1 \}$ is $\neq 0$. For $x \in V_1$, $t_1 \in G_1$, $t_2 \in G_2$ we have $\rho(t_1)\rho(t_2)x = \rho(t_2)\rho(t_2^{-1}t_1t_2)x = \rho(t_2)x$. Hence $\rho(t_1) : V_1 \to V_1$ and this is a rational homom of $G_2/G_1 = K$ to $GL(V_1)$. By the previous lemma, the space $V_2 = \{ v \in V_1 ; \rho(t)v = v, \forall\, t \in G_2 \}$ is $\neq 0$. Continuing we find that $\{ v \in K^m ; \rho(t)v = v, \forall t \in U \}$ is $\neq 0$. $\square$

5) Lemma.

Let $\rho : \underbrace{K^* \times \cdots \times K^*}_{n} \to GL_m(K)$ be a rational homomorphism. (For any k, l in $1, \ldots, m$, the function $(x_1, \ldots x_n) \to \rho(x_1, \ldots x_n)_{kl}$ is a polynomial in x_i, x_i^{-1}.) Then there is a direct sum decomposition $K^m = \bigoplus_\alpha V_\alpha$ $\alpha = (\alpha_1, \ldots \alpha_n) \in \mathbb{Z}^n$ where

$$\rho(x_1, \ldots x_n)|_{V_\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \times \text{Identity}.$$

Proof. It is easy to reduce this to the case $n = 1$. This follows from a previous lemma.

Let

$B = \{ (x_{ij}) \in GL_n(K);\ x_{ij} = 0 \text{ for } i > j \}$. Let $\rho : B \to GL_m(K)$ be a homom. such that for any k, l in $1, \ldots, m$, $g \to \rho(g)_{k,l}$ is a polynomial in x_{ij} $(i \leq j)$ and $(x_{11} x_{22} \cdots x_{nn})^{-1}$.

Theorem (Lie-Kolchin). Assume that $m > 0$. Then there exists $v \in K^m - \{0\}$ and $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ such that

$$\rho(x_{ij}) \cancel{v} = x_{11}^{\alpha_1} x_{22}^{\alpha_2} \cdots x_{nn}^{\alpha_n} v, \quad \forall (x_{ij}) \in B.$$

6) Proof. We have $U \subset B$. By a previous lemma we have $V = \{x \in K^m ;\ \rho(u)x = x,\ \forall\, u \in U\} \neq 0$. Since U is normal in B, V is stable under B and in particular under the subgroup of B consisting of diagonal matrices. Let $x \in V - \{0\}$ be such that x belongs to one of the pieces in the decomposition $V = \bigoplus V_\alpha$ described in an earlier lemma. This satisfies the requirement.

In particular for any rational homomorphism $\rho : GL_n(K) \to GL_m(K)$, $m > 0$, there exists $x \in K^m - \{0\}$ and $(\alpha_1, \ldots \alpha_n) \in \mathbb{Z}^n$ such that

$$\rho\,(x_{ij})\,x = x_{11}^{\alpha_1} \cdots x_{nn}^{\alpha_n}\, x \text{ for all } (x_{ij}) \in B.$$

Such an x is called a highest weight vector of ρ.

Let $B^- = \{ (x_{ij}) \in GL_n(K) \mid x_{ij} = 0 \text{ if } i < j \}$.

Proposition. Let $\rho : GL_n(K) \to GL(V)$ be an irreducible rational homomorphism. Then there is a unique line $L \subset V$ which is B^--stable.

(Irreducible means: $V \neq 0$ and there is no subspace of V other than $0, V$ which is $GL_n(K)$-stable.)

7) The existence follows from Lie-Kolchin applied to B^- instead of B. By the same result applied to $\rho^*: GL_n(K) \to V^*$, where $(\rho^*(g)\xi)v = \xi(\rho(g^{-1})v)$ for $v \in V$, $\xi \in V^*$, we can find $w \in V^* - 0$ such that $\rho(x_{ij})w = x_{11}^{\alpha_1} \cdots x_{nn}^{\alpha_n} w$ for $(x_{ij}) \in B$. Here $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$. Define $F: GL_n(K) \to V^*$ by $F(g) = gw$. Define a homomorphism

$\varphi: V = Hom(V^*, K) \to Reg(GL_n(K))$

by $v \to \varphi(v) =$ composition $[GL_n(K) \xrightarrow{F} V^* \xrightarrow{v} K]$.

φ commutes with the action of $GL_n(K)$, which acts on $Reg(GL_n(K))$ by $(gf)(g_1) = f(g^{-1}g_1)$ i.e. by left translation. We have $\varphi \neq 0$. otherwise we would have $F = 0$ hence $F(1) = w = 0$, absurd. Since V is irreducible, we see that φ is injective. If $v \in V$ then

$\varphi(v)(g(x_{ij})) = v(Fg(x_{ij})) = v(g(x_{ij})w) =$
$x_{11}^{\alpha_1} \cdots x_{nn}^{\alpha_n} v(gw) = x_{11}^{\alpha_1} \cdots x_{nn}^{\alpha_n} \varphi(v)(g)$

for $g \in GL_n(K)$, $(x_{ij}) \in B$. Thus $\varphi(V) \subset A_\alpha$ where

$$A_\alpha = \{f \in Reg(GL_n(K));\ f(g(x_{ij})) = x_{11}^{\alpha_1} \cdots x_{nn}^{\alpha_n} f(g),\ \forall g \in GL_n(K),\ (x_{ij}) \in B\}.$$

Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ can be any element of $\mathbb{Z}^n$.

8) To conclude the proof it is enough to prove:

<u>Lemma.</u> Let $\alpha=(\alpha_1,\dots,\alpha_n)\in \mathbb{Z}^n$. Let
$A_\alpha^{U^-}=\{f\in A_\alpha \mid f(ug)=f(g), \forall g\in G, u\in U^-\}$.
Then $\dim A_\alpha^{U^-}\leq 1$.

Here $U^-=\{(x_{ij})\in GL_n(K);\ x_{ij}=0 \text{ for } i<j,\ x_{ii}=1\ \forall i\}$.

<u>Proof.</u> Define $\lambda: A_\alpha^{U^-}\to K$ (a linear map) by $\lambda(f)=f(1)$. It is enough to prove that λ is injective. Assume that $f\in A_\alpha^{U^-}$, $f(1)=0$. Then $f(ub)=0$ for any $u\in U^-$, $b\in B$. If $y=(y_{ij})\in GL_n(K)$ satisfies $(*)$ $y_{11}\neq 0$, $\begin{vmatrix} y_{11} & y_{12}\\ y_{21} & y_{22}\end{vmatrix}\neq 0$, etc then $y=ub$ for a unique $u\in U^-$, $b\in B$. (exercise)
Thus $f(y)=0$ for any y satisfying $(*)$.
It follows that $f=0$. $\square$

[we use "Weyl's principle of irrelevance of algebraic inequalities": If $P[x_1,\dots,x_n]$ is a polynomial$/K$ which vanishes on
$\{(x_1,\dots,x_n)\in K^n;\ R_1(x_1\dots x_n)\neq 0,\dots R_k(x_1,\dots,x_n)\neq 0\}$ where $R_1,\dots R_k$ are $\neq 0$ polyn.$/K$, then $P=0$. <u>Proof.</u> If $P\neq 0$ then $PR_1\dots R_k\neq 0$ and vanishes on all of K^n. Hence it is 0.]

9) Let $\rho: GL_n(K) \to GL(V)$ be an irred. rat. repres.
We associate to ρ a sequence $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$
as follows:
$\rho(x_{ij})\, v = x_{11}^{\beta_1} \cdots x_{nn}^{\beta_n}\, v$ for any $(x_{ij}) \in B^-$
where $v \in V - 0$ is the unique U^--invariant vector in V (up to nonzero scalar). Note that $\beta = \beta(V)$ depends only on the isomorphism class of V.
For example, assume that $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{Z}^n$ satisfies $\alpha_1 \geqslant \alpha_2 \geqslant \dots \geqslant \alpha_n$. Define $f_\alpha : GL_n(K) \to K$ by

$$f_\alpha(x_{ij}) = x_{11}^{\alpha_1 - \alpha_2} \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}^{\alpha_2 - \alpha_3} \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}^{\alpha_3 - \alpha_4} \cdots \, \|x_{ij}\|^{\alpha_n}$$

Let M_α be the subspace of $Reg(GL_n(K))$ spanned by the left translates of f_α. We have $M_\alpha \subset A_\alpha$ since $f_\alpha \in A_\alpha$ (it is the unique U^- invar. line in A_α).
We have $\dim M_\alpha < \infty$ (see Last Remark.) We have $M_\alpha \neq 0$. We show that M_α is irreducible as repres. of $GL_n(K)$. Assume $M' \subset M_\alpha$ is a non-zero $GL_n(K)$-stable subspace of M'. By an earlier result, M' contains a unique B^- stable line which is necessarily the unique U^--invar. line in A_α. Hence $M' = M_\alpha$

10)

M_α is irreducible. We see that $M_\alpha \neq 0$ for any α such that $\alpha_1 \geq \dots \geq \alpha_n$. We have

$$f_\alpha\left(g\begin{pmatrix} x_1^{-1} & & 0\\ & \ddots & \\ 0 & & x_n^{-1}\end{pmatrix}\right) = x_1^{-\alpha_1}\cdots x_n^{-\alpha_n} f_\alpha(g) \quad \text{for } g\in G,\ x_1,\dots,x_n \in K^*.$$

Hence

$$\begin{pmatrix} x_1 & & 0\\ & \ddots & \\ * & & x_n\end{pmatrix} f_\alpha = x_1^{-\alpha_1}\cdots x_n^{-\alpha_n} f_\alpha .$$

Lemma. Let $(\alpha_1,\dots,\alpha_n)\in \mathbb{Z}^n$ be such that $A_\alpha \neq 0$. Then $\alpha_1\geq\dots\geq\alpha_n$.

Proof. By Last Remark, can find $M\subset A_\alpha$, $\dim M<\infty$, M rat. repres.
Hence exists $f\in A_\alpha - 0$ such that $f(ug)=f(g)$ $\forall u\in U^-$.
Hence $f(ub) = x_{11}^{\alpha_1}\cdots x_{nn}^{\alpha_n} f(1)$ for any $u\in U^-$, $b\in B$.
If $f(1)=0$ then by an earlier argument $f=0$, absurd.
Thus $f(1)\neq 0$. Can assume $f(1)=1$. Now

$$\begin{array}{cc} f\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & t & -1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} & \leftarrow i \\ \uparrow i & \end{array}$$

is a polynomial in t.

Also $\begin{pmatrix} t & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix} 1 & 0\\ t^{-1} & 1\end{pmatrix}\begin{pmatrix} t & -1\\ 0 & t^{-1}\end{pmatrix}$ hence

$$\| $$

$$t^{\alpha_i-\alpha_{i+1}}$$

so that $\alpha_i \geq \alpha_{i+1}$.

<u>Theorem</u>. There is 1-1 correspondence

$\{$iso. classes of irred. rat. repres. of $GL_n(K)\}$

$\Updownarrow$ $\{$sequences $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$ s.t. $\beta_1 \leq \beta_2 \leq \dots \leq \beta_n\}$.

This is obtained by associating to V the unique seq (β_i) such that $\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ * & & x_n \end{pmatrix} v = x_1^{\beta_1} \cdots x_n^{\beta_n} v$ where $v \neq 0$ is the unique U^--invar. vector (up to scalar). In the opposite direction to β we associate the unique irred. rat. subrepres. of A_α where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ satisfies $(\beta_1, \dots, \beta_n) = -(\alpha_1, \dots, -\alpha_n)$.

<u>Last Remark.</u> If $f \in Reg(GL_n(K))$, let M be the subspace of $Reg(GL_n(K))$ spanned by all left translates of f. Then $\dim M < \infty$ and the action of $GL_n(K)$ on M by left translation is a rational repres. of M.

<u>Proof</u>. Easy and omitted.