1)

<u>Representations of $GL_2$</u> (after Frobenius)

Let $V$ be a vector space of dim 2 / finite field $F_q$; let $G = GL(V)$. We describe the conjugacy classes in $G$. Let $F_{q^2}$ be a field with $q^2$ elements containing $F_q$. Let $g \in G$. The eigenvalues of $g: V \to V$ are roots a polynomial of degree 2 with coefficients hence they are $\lambda_1, \lambda_2$ where either (1) $\lambda_1, \lambda_2$ are in $F_q^* = F_q - \{0\}$ or
(2) $\lambda_1, \lambda_2$ are in $F_{q^2} - F_q$ and $\lambda_1^q = \lambda_2$.
If $\lambda_1 \neq \lambda_2$ then the conjugacy class of $g$ is completely determined by $\lambda_1, \lambda_2$. (In case (2) we have automatically $\lambda_1 \neq \lambda_2$.) If $\lambda_1 = \lambda_2$ (so they are in $F_q^*$) then there are two conjugacy classes giving rise to $\lambda_1 = \lambda_2 = \lambda$: one of them is $g = \lambda 1$, the other one is the conjugacy class of $\lambda \cdot u$ where $u - 1 \neq 0$, $(u-1)^2 = 0$. (In some basis, $u$ is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.) Hence #(conj. classes in $G$) =

$$\#\{(\lambda_1, \lambda_2) \in F_q^* \times F_q^*;\ \lambda_1 \neq \lambda_2, \text{ up to order}\}$$
$$+ \#\{\lambda \in F_{q^2} - F_q \text{ up to the equivalence } \lambda \sim \lambda^q\}$$
$$+ 2\, \#\{\lambda \in F_q^*\} = \frac{(q-1)(q-2)}{2} + \frac{q^2-q}{2} + 2(q-1)$$
$$= q^2 - 1.$$

2/

We identify $G$ with the group $GL_2(F_q)$ : $2\times 2$ matrices with entries in $F_q$, $\det \neq 0$. Let $B_0$ be the subgroup $\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| \begin{array}{l} a \in F_q^*, b \in F_q \\ d \in F_q^* \end{array}\right\}$. For any two homomorphisms $\chi_1 : F_q^* \to \mathbb{C}^*$, $\chi_2 : F_q^* \to \mathbb{C}^*$ we define a homomorphism $[\chi_1, \chi_2] : B \to \mathbb{C}^*$ by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \to \chi_1(a)\chi_2(d)$ and we form $I[\chi_1, \chi_2] = \mathrm{Ind}_{B_0}^{G} [\chi_1, \chi_2]$. We compute its character $f = f[\chi_1, \chi_2]$ at $g \in G$ with eigenvalues $\lambda_1, \lambda_2$. If $\lambda_1 \in F_{q^2} - F_q$ then $g$ is not conjugate to an element of $B_0$, hence $f(g) = 0$. If $\lambda_1 \neq \lambda_2$ are in $F_q$ then $g$ is conjugate to each of $\begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_2 & b \\ 0 & \lambda_1 \end{pmatrix}$ $(b \in F_q)$ and the number of $x \in G$ which conjugate $g$ to one of these is $\#\{y \in G \mid yg = gy\} = (q-1)^2$; hence

$$f(g) = \underbrace{\frac{(q-1)^2}{\# B_0} q}_{=1} \left(\chi_1(\lambda_1)\chi_2(\lambda_2) + \chi_1(\lambda_2)\chi_2(\lambda_1)\right)$$

If $\lambda_1 = \lambda_2$ and $g$ is not scalar then $g$ is conjugate to each of $\begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_1 \end{pmatrix}$, $b \in F_q^*$ and the number of $x \in G$ which conjugate $g$ to one of these is $\#\{y \in G \mid yg = gy\} = (q-1)q$; hence $f(g) = \underbrace{\frac{(q-1)^2 q}{\# B_0}}_{=1} \chi_1(\lambda_1)\chi_2(\lambda_1)$.

3)

If $\lambda_1 = \lambda_2$ and $g$ is a scalar then

$$f(g) = \frac{|G|}{|B_0|} \chi_1(\lambda_1)\chi_2(\lambda_1).$$

Now let $\chi_1', \chi_2'$ be another pair of characters of $F_q^*$ and let $f' = f[\chi_1', \chi_2']$. We compute the inner product $(f|f')$. By Frobenius reciprocity this equals

$$\frac{1}{\# B_0} \sum_{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B_0} \overline{f\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}} \chi_1'(a)\chi_2'(d) =$$

$$= \frac{q}{\# B_0} \sum_{\substack{a \neq d \\ \text{in } F_q^*}} \left(\chi_1(a)\chi_2(d) + \chi_2(a)\chi_1(d)\right) \overline{\chi_1'(a)\chi_2'(d)}$$

$$+ \frac{q-1}{\# B_0} \sum_{a \in F_q^*} \chi_1(a)\chi_2(a) \overline{\chi_1'(a)\chi_2'(a)}$$

$$+ \frac{q+1}{\# B_0} \sum_{a \in F_q^*} \chi_1(a)\chi_2(a) \overline{\chi_1'(a)\chi_2'(a)}$$

$$= \frac{1}{(q-1)^2} \sum_{\substack{a, d \\ \text{in } F_q^*}} \left(\chi_1(a)\overline{\chi_1'(a)}\, \chi_2(d)\overline{\chi_2'(d)} + \chi_2(a)\overline{\chi_1'(a)}\, \chi_1(d)\overline{\chi_2'(d)}\right)$$

$$= \delta_{\chi_1, \chi_1'} \delta_{\chi_2 \chi_2'} + \delta_{\chi_2 \chi_1'} \delta_{\chi_1, \chi_2'}.$$

4) Thus

$$(f|f') = \begin{cases} 2 & \text{if} \quad \chi_1 = \chi_1' = \chi_2 = \chi_2' \\ 1 & \text{if} \quad \chi_1 \neq \chi_2,\ \chi_1' \neq \chi_2' \text{ and} \\ & (\chi_1, \chi_2) = (\chi_1', \chi_2') \text{ up to order} \\ 0 & \text{otherwise.} \end{cases}$$

We see that

$I(\chi_1, \chi_2)$ depends only on the unordered pair $(\chi_1, \chi_2)$;

$I(\chi_1, \chi_2)$ is irreducible if $\chi_1 \neq \chi_2$ and is non-isomorphic to $I(\chi_1', \chi_2')$ if $\chi_1' \neq \chi_2'$ are not the same as $\chi_1, \chi_2$ up to order;

$I(\chi_1, \chi_2)$ with $\chi_1 = \chi_2$ is the direct sum of two non-isomorphic irred repres. and is disjoint from $I(\chi_1', \chi_2')$ if $(\chi_1', \chi_2')$ is not the same as $(\chi_1, \chi_2)$.

The number of distinct irred. reps. obtained in this way is $\dfrac{(q-1)(q-2)}{2} + 2(q-1)$.

This is short of the expected number $q^2-1$. The remaining irred. rep. are more difficult to get. Frobenius obtained them as differences of induced repres.

5) We identify $V = F_{q^2}$ as an $F_q$-vector space. Then $T = F_{q^2}^*$ can be viewed as a subgroup of $G$. (Left multiplication by an element of $F_{q^2}^*$ is an $F_q$-linear isomorphism $V \xrightarrow{\sim} V$.) Let $\chi : F_{q^2}^* \to \mathbb{C}^*$ be a homomorphism. We can form $\mathrm{Ind}_T^G(\chi)$. Let $f_\chi$ be its character.

We compute $f_\chi(g)$ for $g \in G$ with eigenvalues $\lambda_1, \lambda_2$. An element $\lambda \in F_{q^2}^* = T$, viewed as element of $G$ has eigenvalues $\lambda, \lambda^q$. Hence if $\lambda_1, \lambda_2$ are distinct in $F_q^*$ then $f_\chi(g) = 0$.

If $\lambda_1, \lambda_2$ are in $F_{q^2} - F_q$ then $g$ is conjugate ~~two~~ exactly two elements of $T$, represented by $\lambda_1, \lambda_2$ and the number of $x \in G$ which conjugate $g$ to one of these elements equals $\#\{y \in G \mid yg = gy\} = \# T = q^2 - 1$. Thus

$$f_\chi(g) = \underbrace{\frac{\# T}{\# T}}_{=1} \left( \chi(\lambda_1) + \chi(\lambda_2) \right).$$

If $\lambda_1 = \lambda_2$ then $g$ must be a scalar and

$$f_\chi(g) = \frac{\# G}{\# T} \chi(\lambda_1) = q(q-1)\, \chi(\lambda_1).$$

6) Let $H$ be the subgroup $\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}; \begin{matrix} a \in F_q^* \\ b \in F_q \end{matrix}\right\}$ of $B_0$.

We fix a nontrivial homomorphism $\psi : F_q \to \mathbb{C}^*$. Recall that $\chi : F_{q^2}^* \to \mathbb{C}^*$ was given. We define $\tilde{\chi} : H \to \mathbb{C}^*$ by $\tilde{\chi}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \chi(a)\,\psi\!\left(\frac{b}{a}\right)$. We have $\tilde{\chi}\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix}\right)$

$$= \tilde{\chi}\begin{pmatrix} aa' & ab'+ba' \\ 0 & aa' \end{pmatrix} = \chi(aa')\,\psi\!\left(\frac{ab'+ba'}{aa'}\right)$$

$$= \chi(aa')\,\psi\!\left(\frac{b}{a}\right)\psi\!\left(\frac{b'}{a'}\right) = \tilde{\chi}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\tilde{\chi}\begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix}.$$

Thus, $\tilde{\chi}$ is a homomorphism. Let $f'_\chi : G \to \mathbb{C}$ be the character of $\mathrm{Ind}_H^G(\tilde{\chi})$. We compute $f'_\chi(g)$ for $g \in G$ with eigenvalues $\lambda_1, \lambda_2$. If $\lambda_1 \neq \lambda_2$ then $g$ is not conjugate to an element of $H$, so that $f'_\chi(g) = 0$. If $\lambda_1 = \lambda_2$ and $g$ is non-scalar then $g$ is conjugate to each of $\begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_1 \end{pmatrix}$ with $b \neq 0$ and the number of elements which conjugate $g$ to such an element is $\#\left\{y \in G \,/\, y\begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_1 \end{pmatrix} y\right\} = q(q-1)$. Hence

$$f'_\chi(g) = \frac{q(q-1)}{\# H}\sum_{b \in F_q^*} \chi(\lambda_1)\,\psi\!\left(\frac{b}{\lambda_1}\right) = -\chi(\lambda_1).$$

If $\lambda_1 = \lambda_2$ and $g$ is scalar then $f'_\chi(g) = \frac{\# G}{\# H}\chi(\lambda_1) = (q^2-1)\,\chi(\lambda_1)$.

7)

We now consider the difference $f'_\chi - f_\chi$. Its value at $g \in G$ with eigenvalues $\lambda_1, \lambda_2$ is

$$\begin{array}{ll} -(\chi(\lambda_1) + \chi^q(\lambda_2)) & \text{if } \lambda_1 \in F_{q^2} - F_q \\ 0 & \text{if } \lambda_1 \neq \lambda_2 \text{ in } F_q \\ -\chi(\lambda_1) & \text{if } \lambda_1 = \lambda_2,\ g \text{ non scalar} \\ (q-1)\chi(\lambda_1) & \text{if } \lambda_1 = \lambda_2,\ g \text{ scalar.} \end{array}$$

We now consider another character $\chi' : F_{q^2}^\times \to \mathbb{C}^*$ and we compute $(f'_\chi - f_\chi \mid f'_{\chi'} - f_{\chi'})$. It is

A

$$\frac{1}{\# G}\left( \sum_{\lambda \in F_{q^2} - F_q} (\chi(\lambda) + \chi(\lambda^q)) \overline{(\chi'(\lambda) + \chi'(\lambda^q))} \cdot \frac{N}{2} \right.$$

$$\left. + \sum_{\lambda \in F_q^*} \chi(\lambda)\overline{\chi'(\lambda)} \cdot N' + \sum_{\lambda \in F_q^*} (q-1)^2 \chi(\lambda)\overline{\chi'(\lambda)} \right)$$

where $N = \dfrac{\# G}{(q-1)^2}$, $N' = \dfrac{\# G}{q(q-1)}$. Thus

$$A = \frac{1}{(q-1)^2}\left( \sum_{\lambda \in F_{q^2} - F_q} \chi(\lambda)\overline{\chi'(\lambda)} + \sum_{\lambda \in F_{q^2} - F_q} \chi(\lambda)\overline{\chi'^q(\lambda)} + 2 \sum_{\lambda \in F_q^*} \chi(\lambda)\overline{\chi'(\lambda)} \right)$$

$$= \frac{1}{(q-1)^2}\left( \sum_{\lambda \in F_{q^2}^*} \chi(\lambda)\overline{\chi'(\lambda)} + \sum_{\lambda \in F_{q^2}^*} \chi(\lambda)\overline{\chi'^q(\lambda)} \right)$$

$$= \delta_{\chi, \chi'} + \delta_{\chi, \chi'^q}.$$ Thus

8) $(f'_\chi - f_\chi \mid f'_{\chi'} - f_{\chi'}) = \begin{cases} 2 & \text{if } \chi = \chi' = \chi^q \\ 1 & \text{if } \chi \neq \chi^q \text{ and } \\ & \chi' \text{ is } \chi \text{ or } \chi^q \\ 0 & \text{otherwise} \end{cases}$

We see that $f'_\chi - f_\chi$ is the char. of an irred. rep. of degree $q+1$ whenever $\chi \neq \chi^q$, which is not isom. to the char. associated to $\chi'$ with $\chi' \neq \chi'^q$, $\chi' \notin \{\chi, \chi^q\}$.

We thus obtain $\frac{q^2-q}{2}$ distinct irred. rep. of degree $q+1$, which must be distinct from the previously constructed repres. which have degree $\leq q$. The total number of distinct irr. reps. constructed above is

$$\frac{(q-1)(q-2)}{2} + 2(q-1) + \frac{q^2-q}{2} = q^2-1.$$

We see that all irred. reps. of $G$ are obtained exactly once.

Now assume that $\dim V/\mathbb{F}_q$ is 3. The irr. reps. of $GL(V)$ have the following degrees. (Steinberg)

9)

$1$ ——— 1 param

$q^2+q$ ——— "

$q^3$ ---- Steinberg $\otimes$ 1-dim. ——— "

$(q+1)(q^2+q+1)$ ----- induced from ——— (3 param)

$q^2+q+1$ ——— (2 param)

$q(q^2+q+1)$ ——— (2 param)

$(q-1)(q^2+q+1)$ ----- induced from ——— (3 param)

$(q-1)(q^2-1)$ ----- these are the most difficult (3 param)

The irred rep of $GL_n(\mathbb{F}_q)$ were classified (and their character determined) by J. A. Green (1955)