1)

Introduction to representation theory.

TOPICS:

1) Representations of finite groups, see; Serre - Linear representations of finite groups [uses only linear algebra].
2) Representations of the symmetric group (following Specht 1932)
3) Bruhat decomposition in GL_n, Hecke algebra
4) Repres. of GL_2 over a finite field (following Frobenius)
5) Repres of GL_n over a finite field appearing in functions on flag manifolds. Steinberg representation
6) Modular representations of GL_n over a finite field, following Carter-Lusztig 1976.
7) Rational representations of GL_n following Chevalley
8) The new basis of a Hecke algebra
9) Weyl character formula and p-analogue

2) Linear representations.

V : vector space / $\mathbb{C}$

$GL(V) = \{T : V \to V$ linear isomorphism$\}$: is a group under composition. If V has basis $\{e_i \mid i = 1, \dots, n\}$, then $GL(V)$ can be identified with set of $n \times n$ matrices (a_{ij}) $a_{ij} \in \mathbb{C}$ such that $\det(a_{ij}) \neq 0$, by:

$T \to (a_{ij})$ where $T(e_j) = \sum_i a_{ij} e_i$.

composition corresponds to matrix multiplication.

G : finite group. A linear representation of G in V is a homomorphism

$\rho : G \to GL(V)$.

Thus to each $s \in G$ we associate $\rho(s) \in GL(V)$ so that $\rho(st) = \rho(s)\rho(t)$ for $s, t \in G$. It follows that $\rho(1) = 1$, $\rho(s^{-1}) = \rho(s)^{-1}$.

We always assume $\dim V < \infty$. We say $\dim V$ is the degree of ρ.

3) If (e_i) is a basis of V we have

$$\rho(s)(e_j) = \sum_i r_{ij}(s) e_i \quad \text{and}$$

$$r_{ik}(st) = \sum_j r_{ij}(s) r_{jk}(t), \ \forall i,k.$$

If ρ, ρ' are repres. of G in V, V', we say that ρ, ρ' are isomorphic if there exists a linear isom. $\tau : V \to V'$ such that

$$\begin{tikzcd} V \arrow[r, "\rho(s)"] \arrow[d, "\tau"'] & V \arrow[d, "\tau"] \\ V' \arrow[r, "\rho'(s)"] & V' \end{tikzcd}$$

is commutative ($\tau \rho(s) = \rho'(s) \tau$)

for any $s \in G$.

Examples. (1) A repres. of degree 1 of G is a homom $\rho : G \to \mathbb{C}^* = \mathbb{C} - \{0\}$.
The unit repres. of G is $\rho : G \to \mathbb{C}^*$, $\rho(s) = 1$, $\forall s$.
(2) Let $g = \#(G)$, V a $\mathbb{C}$ vector space with basis $\{e_t \,;\, t \in G\}$. For $s \in G$ define lin. iso. $\rho(s) : V \to V$ by $\rho(s) e_t = e_{st}$. This is a linear repres., the "regular representation" of G.

4) Assume X is a finite set on which G acts. Thus for any $s \in G$ we are given a bijection $x \to sx$ from X to X such that $s(tx) = (st)x \quad \forall s, t, x$ and $1x = x \; \forall x$

Let V be a vector space with basis $\{e_x \, ; x \in X\}$. For $s \in G$ define $\rho(s) : V \to V$ (lin. isom.) by $\rho(s)(e_x) = e_{sx} \quad \forall x \in X$.

This is a linear repres. of G.

(The regular repres. is a special case.)

Let $\rho : G \to GL(V)$ be a lin. repres. A subspace $W \subset V$ is invariant (or stable) if $\rho(s)x \in W$ for any $x \in W$. Let $\rho^W(s) : W \to W$ be the restriction of $\rho(s)$. Then $s \to \rho^W(s)$ is a lin. repres. of G on W. We show:

<u>Lemma.</u>

For W as above there exists a subspace $W' \subset V$ such that W' is invariant and $V = W \oplus W'$.

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Proof. We can find a subspace $W_0' \subset V$ (perhaps non-invariant) such that $V = W \oplus W_0'$. Let $T_0 : V \to V$ be the unique linear map such that $T_0|_{W_0'} = 0$, $T_0|_W = \text{identity}$. We define

$T : V \to V$ by $T(x) = \frac{1}{\#G} \sum_{t \in G} \rho(t) T_0 \, \rho(t^{-1})(x)$

We have $\rho(s) T = T \rho(s)$ for any s:

$\sum_t \rho(st) T_0 \, \rho(t^{-1}) \overset{?}{=} \sum_t \rho(t) T_0 \, \rho(t^{-1} s)$

$= \sum_{t'} \rho(t') T_0 \, \rho(t'^{-1} s)$.

If $x \in V$ we have $\rho(t) T_0 \, \rho(t^{-1})(x) \in \rho(t) W = W$ hence $TV \subset W$. If $x \in W$ we have $\rho(t) T_0 \, \rho(t^{-1})(x) = \rho(t) \rho(t^{-1}) x = x$ since $\rho(t^{-1}) x \in W$ and $T_0|_W = 1$.

Hence $T(x) = \frac{1}{\#G} \sum_{t \in G} x. = x.$

6/ Let $W' = \{x \in V \mid Tx = 0\}$. This is an invariant subspace of V since $T\rho(s) = \rho(s)T$. We have $W \cap W' = 0$. (If $x \in W \cap W'$ then $Tx = x$ and $Tx = 0$, so $x = 0$). We have $V = W + W'$. (If $x \in V$ then $x = Tx + (x - Tx)$ with $Tx \in W$, $T(x - Tx) = Tx - Tx = 0$ since $T^2 = T$ hence $x - Tx \in W'$.) Thus, $V = W \oplus W'$. $\square$

If $\rho_1 : G \to GL(W_1)$, $\rho_2 : G \to GL(W_2)$ are lin. repres., then $\rho : G \to GL(W_1 \oplus W_2)$ where $\rho(s) : W_1 \oplus W_2 \to W_1 \oplus W_2$ is $(w_1, w_2) \to (\rho_1(s) w_1, \rho_2(s) w_2)$ is a lin. repres. (the "direct sum" of ρ_1, ρ_2). Similarly we can define the direct sum of the lin. repres $\rho_i : G \to GL(W_i)$, $i = 1, \dots, k$.
In the previous lemma, the lin. rep. V is isom. to the direct sum of the rep. on W and the one on W'.

7/ Let $\rho : G \to GL(V)$ be a lin-rep.
We say that ρ is irreducible (or simple) if $V \neq 0$ and there is no invariant subspace $W \subset V$ other than $0, V$.

|| Any lin. rep. is a direct sum of irred. reps. (follows from Lemma a.)

If $\rho_1 : G \to GL(V_1)$, $\rho_2 : G \to GL(V_2)$ are lin. repres. then

$$\rho : G \to GL(V_1 \otimes V_2)$$

$$\rho(s)(x \otimes y) = \rho_1(s)x \otimes \rho_2(s)y$$

for $x \in V_1$, $y \in V_2$, $s \in G$
is a lin. repres. of degree $\dim(V_1)\dim(V_2)$.

If $\rho : G \to GL(V)$ is a lin. rep. and V^* is the vector space dual to V then

$$\rho^* : G \to GL(V^*), \quad \rho^*(s) = \rho(s^{-1})^{transpose}$$

is a lin. rep. (If $T : V \to V$ is a lin. map, then $T^{transp} : V^* \to V^*$ is $\xi \to \xi'$ where $\xi'(v) = \xi(T(v))$ for $v \in V$.)

8)

Grothendieck group.

Let $R(G)$ be the free abelian group with basis given by the irred. rep. of G (up to isomorphism). If $\rho: G \to GL(V)$ is a repres. of G, then ρ can be viewed as the element $\sum_i m_i \rho_i$ where ρ_i are the irred. reps. of G and $m_i = (\rho_i ; \rho)$ are integers ≥ 0. The direct sum of two representations corresponds to the sum in $R(G)$ of the elements corresp. to the two representations.