A construction of character sheaves on parahoric subgroups

Lie groups seminar, MIT

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Content

- Classical constructions
- Parahoric/deep level constructions
- Relation with supercuspdal representations
- An alternative construction

The finite setting

Let $p \neq \ell$ be two prime numbers, and $\mathbb{F}_p$ the prime field.

- $\mathbb{F}_q$ a finite extension of $\mathbb{F}_p$;
- $G = G(\overline{\mathbb{F}}_q)$ a reductive group over $\mathbb{F}_q$;
- F the Frobenius automorphism of G;
- $G^F = G(\mathbb{F}_q)$ a finite group of Lie type;
- $T \subseteq G$ a maximal torus over $\mathbb{F}_q$;
- $B = TU$ a Borel subgroup with unipotent radical U.

Deligne-Lusztig representations

- For $B = TU$ the Deligne-Lusztig variety is

$$Y = \{g \in G; g^{-1}F(g) \in FU\}.$$

- The group $G^F \times T^F$ acts on Y by left/right multiplication. Thus

$$H_c^i(Y, \overline{\mathbb{Q}}_\ell) = \bigoplus_{\theta \in \widehat{T^F}} H_c^i(Y, \overline{\mathbb{Q}}_\ell)_\theta.$$

- For a $\theta \in \widehat{T^F}$, the virtual G^F-module

$$R_T^G(\theta) = H_c^*(Y, \overline{\mathbb{Q}}_\ell)_\theta := \sum_i (-1)^i H_c^i(Y, \overline{\mathbb{Q}}_\ell)_\theta$$

is called a **Deligne-Lusztig representation**.

Lusztig's character sheaves

- Consider the variety $\widetilde{G} = \{(g, hB) \in G \times G/B; h^{-1}gh \in B\}$.
- There are natural maps

$$T \xleftarrow{f} \widetilde{G} \xrightarrow{\pi} G, \qquad \mathrm{pr}_T(h^{-1}gh) \leftarrow (g, hB) \rightarrow g.$$

- Consider $\mathrm{Ind}_B^G = \pi_! f^*[2 \dim G/B] : D_T(T) \rightarrow D_G(G)$.
- Irreducible constituents of ${}^pH^i(\mathrm{Ind}_B^G(\mathcal{L}))$ for multiplicative $\mathcal{L}$ and $i \in \mathbb{Z}$ are called **character sheaves**.

Properties

Let $\chi_{\mathcal{F}} : G^F \to \overline{\mathbb{Q}}_\ell$ be the Frobenius trace of an F-stable simple perverse sheave $\mathcal{F}$.

Theorem (Lusztig)

- $\mathrm{Ind}_B^G(\mathcal{L}[\dim T])$ *is semisimple perverse for multiplicative* $\mathcal{L}$*;*
- *The Forbenius traces of* F*-stable characters sheaves form a nice basis of class functions on* G^F*;*
- *If* $G = GL_n$*, the Forbenius traces of characters sheaves are irreducible characters of* G^F*;*
- *If* $\theta \in \widehat{T}^F$ *is regular, then* $\mathrm{Ind}_B^G(\mathcal{L}_\theta[\dim T])$ *is simple and*

$$\chi_{\mathrm{Ind}_B^G(\mathcal{L}_\theta[\dim T])} = \pm R_T^G(\theta).$$

Here $\mathcal{L}_\theta$ *is the rank-one multiplicative local system corresponding to* θ*.*

The p-adic setting

Let k be a non-archimedean field with residue field $\mathbb{F}_q$.

- $\breve{k}$ the completion of a maximal unramified extension of k;
- G a reductive group over k;
- $T \subseteq G$ a maximal (elliptic) torus over k, splitting over $\breve{k}$;
- $\mathbf{x} \in \mathcal{B}(G, k) \cap \mathcal{A}(T, \breve{k})$ a fixed point;
- $G_{\mathbf{x},s} \subseteq G_x$ the s-th Moy-Prasad subgroup for $s \in \mathbb{R}_{\geqslant 0}$.
- for $s, r \geqslant 0$ put $G_{s:r} = G_{\mathbf{x},s}/G_{\mathbf{x},r+}$, $T_r = T_{\mathbf{x},0}/T_{\mathbf{x},r+}$ and so on.

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Question

How to construct Deligne-Lusztig representations and character sheaves for G_r?

Parahoric DL representations

Definition (Lusztig)

Let $B = TU \subseteq G$ be a Borel subgroup.

- the **parahoric Deligne-Luszitg variety** is

$$Y_r = \{g \in G_r; g^{-1}F(g) \in FU_r\},$$

  on which $G_r^F \times T_r^F$ acts by left/right multiplication.

- for $\phi \in \widehat{T^F}$ of depth r, the virtual G_r^F-module

$$R_{T,r}^G(\phi) = H_c^*(Y_r, \overline{\mathbb{Q}}_\ell)_\phi$$

  is called a **parahoric Deligne-Lusztig representation**.

The parahoric analogue

- Consider $\widetilde{G}_r = \{(g, hB_r) \in G_r \times G_r/B_r; h^{-1}gh \in B_r\}$.
- There are natural maps

$$T_r \xleftarrow{f} \widetilde{G}_r \xrightarrow{\pi} G_r, \qquad \mathrm{pr}_{T_r}(h^{-1}gh) \leftarrow (g, hB_r) \to g.$$

- Consider $\mathrm{Ind}_{B_r}^{G_r} = \pi_! f^*[2\dim G_r/B_r] : D(T_r) \to D(G_r)$.

Conjecture (Lusztig)

If ϕ is generic of depth r, then $\mathrm{Ind}_{B_r}^{G_r}(\mathcal{L}_\phi[\dim T_r])$ is an intersection cohomology complex .

The parabolic induction/restriction functors

Let $T \subseteq P = LN \subseteq G$ be parabolic subgroup.

Fix $r \in \mathbb{Z}_{\geqslant 1}$. Consider the natural correspondence of stacks

$$L_r/L_r \xleftarrow{\ \alpha\ } P_r/P_r \xrightarrow{\ \beta\ } G_r/G_r.$$

The induction/restriction functors are defined by

$$\mathrm{Ind}_{P_r}^{G_r} = \beta_! \circ \alpha^* : D_{L_r}(L_r) \longrightarrow D_{G_r}(G_r);$$
$$\mathrm{Res}_{P_r}^{G_r} = \alpha_* \circ \beta^! : D_{G_r}(G_r) \longrightarrow D_{L_r}(L_r).$$

(L, G)-generic subcategories

Assume $r > 0$. Set $\mathfrak{g} = G_{r:r}$ and $\mathfrak{l} = L_{r:r}$.

Let $X_\psi \in \mathfrak{l}^*$ be an (L, G)-**generic** function. Consider the Fourier transforms

$$\mathcal{L}_{\psi,r} := FT(\delta_{X_\psi})[\dim \mathfrak{l}] \in D_{L_r}(\mathfrak{l}), \qquad \mathfrak{F}_{\psi,r} = FT(\delta_{G_r \cdot X_\psi})[\dim \mathfrak{g}] \in D_{G_r}(\mathfrak{g}).$$

Here $G_r \cdot X_\psi$ denotes the coadjoint orbit of X_ψ.

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Here $G_r \cdot X_\psi$ denotes the coadjoint orbit of X_ψ.

Definition (Bezrukavnikov-Chan)

The (L, G)-generic subcategories of $D_{L_r}(L_r)$ and $D_{G_r}(G_r)$ are given by

$$D^{\psi}_{L_r}(L_r) := \mathcal{L}_{\psi,r} \star_! D_{L_r}(L_r) = \mathcal{L}_{\psi,r} \star_* D_{L_r}(L_r)$$
$$D^{\psi}_{G_r}(G_r) := \mathcal{F}_{\psi,r} \star_! D_{G_r}(G_r) = \mathcal{F}_{\psi,r} \star_* D_{G_r}(G_r).$$

t-exactness

Theorem (Bezrukavnikov-Chan)

The parabolic induction restricts to a t-exact equivalence

$$\mathrm{Ind}_{P_r}^{G_r} : D_{L_r}^{\psi}(L_r) \xrightarrow{\sim} D_{G_r}^{\psi}(G_r),$$

whose inverse is $\mathcal{L}_{\psi,r} \star_* \mathrm{Res}_{P_r}^{G_r}$.

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whose inverse is $\mathcal{L}_{\psi,r} \star_* \mathrm{Res}_{P_r}^{G_r}$.

Corollary (B.-C.)

In the (T, G)-generic case, consider $T_{r,\mathrm{vreg}} \xleftarrow{f_{\mathrm{vreg}}} \widetilde{G}_{r,\mathrm{vreg}} \xrightarrow{\pi_{\mathrm{vreg}}} G_{r,\mathrm{vreg}} \xhookrightarrow{j} G_r$. *Then*

$$\mathrm{Ind}_{B_r}^{G_r}(\mathcal{L}[\dim T_r]) \cong j_{!*}(\pi_{\mathrm{vreg}!} f_{\mathrm{vreg}}^* \mathcal{L}_{\mathrm{vreg}}[\dim G_r]) \quad \textit{for} \quad \mathcal{L} \in D_{T_r}^{\psi}(T_r).$$

In particular, Lusztig's conjecture is true.

Comparison with Deep level DL characters

Theorem (B.-C.)

Suppose $p, q \gg 0$*. Let* $\phi \in \widehat{T}^F$ *be* (T, G)*-generic of depth* $\leqslant r$*. Then*

$$\chi_{\mathrm{Ind}_{B_r}^{G_r}(\mathcal{L}_\phi[\dim T_r])} = (-1)^{\dim G_r} R_{T_r}^{G_r}(\phi).$$

Question

Does the equality hold true when ϕ *is regular?*

Generic datum

Definition

A (normalized) **generic datum** is a triple $\Lambda = (G^i, \phi_i, r_i)_{0 \leqslant i \leqslant d}$, where

- $G^0 \subsetneq \cdots \subsetneq G^d = G$ are Levi subgroups;
- $0 = r_{-1} < r_0 < \cdots < r_{d-1} \leqslant r_d$ if $1 \leqslant d$ and $0 \leqslant r_0$ if $d = 0$;
- ϕ_i is a character of $(G^i)^F$ and trivial over $(G^i_{sc})^F$ for $0 \leqslant i \leqslant d$;
- ϕ_i is of depth r_i and (G^i, G^{i+1})-**generic** for $0 \leqslant i \leqslant d-1$;
- ϕ_d is of depth r_d if $r_{d-1} < r_d$, and is trivial otherwise;

Yu's datum

Definition (J.-K. Yu)

A **Yu's datum** is a tuple $\Sigma = (\Lambda, \mathbf{x}, \rho)$, where

- $\Lambda = (G^i, \phi_i, r_i)_{0 \leqslant i \leqslant d}$ is a generic datum;
- $\mathbf{x} \in \mathcal{B}(G^0, k)$ whose natural image $\bar{\mathbf{x}}$ in $\mathcal{B}(G^0_{\mathrm{der}}, k)$ is a vertex;
- $Z(G^0)/Z(G)$ is anisotropic;
- ρ is an irr. cuspidal rep'n of the reductive quotient of $(G^0_{\mathbf{x}})^F$.

Here $G^0_{\mathbf{x}}$ is the stabilizer of $\bar{\mathbf{x}}$ in G^0.

Yu's construction

To the above Yu's datum $\Sigma = (\Lambda, \mathbf{x}, \rho)$, one can associate a subgroup

$$\widetilde{K} = \widetilde{K}_\Lambda := G^0_{\mathbf{x}} G^1_{\mathbf{x}, r_0/2} \cdots G^d_{\mathbf{x}, r_{d-1}/2}$$

and the **Weil-Heisenberg rep'n** $\kappa = \kappa_\Lambda$ of $\widetilde{K}^F$

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Theorem (Yu, Kim, Fintzen...)

For almost all primes p the map $\Sigma \mapsto \pi_\Sigma := c\text{-}\mathrm{ind}_{K^F}^{G^F} \widetilde{\kappa} \otimes \rho$ *gives a bijection*

$$\{\textit{generic cuspidal data}\}_{/\cong} \xrightarrow{\sim} \{\textit{irre. supercuspidal rep'ns}\}_{/\cong}.$$

Howe Factorization

Let $\phi \in \widehat{T^F}$ be a smooth character.

Definition

A **Howe factorization** of $\phi \in \widehat{T^F}$ is a pair (Λ, ϕ_{-1}), where

- $\Lambda = (G^i, \phi_i, r_i)_{0 \leqslant i \leqslant d}$ is a generic datum;
- $\phi_{-1} \in \widehat{T}^F$ is a character of depth 0;
- $\phi = \prod_{i=-1}^{d} \phi_i|_{T^F}$.

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Theorem (Kaletha)

Howe factorizations exist if p is good for G and $p \nmid |\pi_1(G_{\mathrm{der}})|$.

Subgroups attached to ϕ

Fix a Howe factorization $(\Lambda, , \phi_{-1})$ of ϕ with $\Lambda = (G^i, \phi_i, r_i)_{0 \leqslant i \leqslant d}$.

- $K_\phi = G^0_{\mathbf{x},0} G^1_{\mathbf{x},r_0/2} \cdots G^d_{\mathbf{x},r_{d-1}/2}$;
- $H = G^0_{\mathbf{x},0+} G^1_{\mathbf{x},r_0/2} \cdots G^d_{\mathbf{x},r_{d-1}/2}$;
- $K^+_\phi = G^0_{\mathbf{x},0+} G^1_{\mathbf{x},r_0/2+} \cdots G^d_{\mathbf{x},r_{d-1}/2+}$;
- $T_\phi = T^0_{0+} T^1_{r_0+} \cdots T^d_{r_{d-1}+}$, where $T^i = G^i_{\mathrm{der}} \cap T$.

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- $H = G^0_{\mathbf{x},0+} G^1_{\mathbf{x},r_0/2} \cdots G^d_{\mathbf{x},r_{d-1}/2}$;
- $K_\phi^+ = G^0_{\mathbf{x},0+} G^1_{\mathbf{x},r_0/2+} \cdots G^d_{\mathbf{x},r_{d-1}/2+}$;
- $T_\phi = T^0_{0+} T^1_{r_0+} \cdots T^d_{r_{d-1}+}$, where $T^i = G^i_{\mathrm{der}} \cap T$.

For $B = TU$ define the Iwahori type subgroups

$$I_\phi = (K_\phi \cap U) T (K_\phi^+ \cap \overline{U});$$
$$I_\phi^\dagger = (K_\phi \cap U) T_\phi (K_\phi^+ \cap \overline{U}),$$

where $\overline{U}$ is the opposite of U.

Variations of deep level constructions

Let $K_{\phi,r}$ be the image of K_ϕ in G_r and so on.

- Consider the following variety

$$Z_{\phi,r} = \{g \in G_r; g^{-1}F(g) \in FI^{\dagger}_{\phi,r}\},$$

  on which $G_r^F \times T_r^F$ acts by left/right multiplication.

- Define the virtual G_r^F-module

$$\mathcal{R}_{T_r}^{G_r}(\phi) = H_c^*(Z_{\phi,r}, \overline{\mathbb{Q}}_\ell)_\phi = \mathrm{ind}_{K_{\phi,r}^F}^{G_r} H_c^*(Z_{\phi,r}^K, \overline{\mathbb{Q}}_\ell)_\phi,$$

  where $Z_{\phi,r}^K = Z_{\phi,r} \cap K_{\phi,r}$.

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  where $Z_{\phi,r}^K = Z_{\phi,r} \cap K_{\phi,r}$.

Remark

The module $\mathcal{R}_{T_r}^{G_r}(\phi)$ is independent of choice of $B \supseteq T$.

Geometric analogue of Weil-Heisenberg representation

Suppose $L := G^0$ is standard w.r.t. $B = TU$, and hence $L_{\mathbf{x}}$ normalizes $H_\phi \cap I_\phi$.

Note that $K_{\phi,r} = H_{\phi,r} L_r$, we put $Z^H_{\phi,r} = Z_{\phi,r} \cap H_{\phi,r}$ and $Z^L_{\phi,r} = Z_{\phi,r} \cap L_r$.

- The action

$$H^F_{\phi,r} \rtimes L^F_r \curvearrowright Z^H_{\phi,r} \times Z^L_{\phi,r}, \quad (h,l) : (x,y) \mapsto (hlxl^{-1}, ly)$$

gives $H^F_{\phi,r} \rtimes L^F_r$-modules $H^i_c(Z^H_{\phi,r}, \overline{\mathbb{Q}}_\ell)_\phi$ and L^F_r-modules $H^i_c(Z^L_{\phi,r}, \overline{\mathbb{Q}}_\ell)_\phi$;

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- $H^i_c(Z^L_{\phi,r}, \overline{\mathbb{Q}}_\ell)_\phi \cong (\prod_{i=0}^d \phi_i) \otimes H^i_c(Z^L_{\phi,0}, \overline{\mathbb{Q}}_\ell)_{\phi_{-1}}$;
- $H^i_c(Z^H_{\phi,r}, \overline{\mathbb{Q}}_\ell)_\phi \otimes (\prod_{i=0}^d \phi_i)$ descend to $K^F_{\phi,r} = H^F_{\phi,r}L^F_r$-modules.

Geometric analogue of Weil-Heisenberg representation

Put $\kappa_\phi = H_c^*(Z_{\phi,r}^H, \overline{\mathbb{Q}}_\ell)_\phi \otimes (\prod_{i=0}^d \phi_i)$ as a virtual $K_{\phi,r}^F$-module.

Theorem (N., Liu-N.)

We have the following properties:

- $R\Gamma_c(Z_{\phi,r}^H, \overline{\mathbb{Q}}_\ell)_\phi$ *is nonzero at a single degree;*
- $\pm\kappa_\phi \cong \kappa|_{K_{\phi,r}^F} \otimes \varepsilon$ *for some character* ε *of* $L_{\mathbf{x},0}^F$*;*
- *if* $q > 3$, $\varepsilon = \epsilon|_{L_0^F}$ *with* ϵ *the quadratic character of* $L_{\mathbf{x}}^F$ *by Fintzen-Kaletha-Spice.*

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Remark

The results also hold in mod ℓ coefficient case.

Cohomology of $Z^K_{\phi,r}$

Proposition (N., Ivanov-N.)

The map $(x,y)\mapsto xy$ *gives an equivariant* $L^F_{0+:r}$*-torsor*

$$f : Z^H_{\phi,r}\times Z^L_{\phi,r}\to Z^K_{\phi,r},$$

which induces an isomorphism of $K^F_{\phi,r}$*-modules*

$$\begin{aligned} R\Gamma_c(Z^K_{\phi,r},\overline{\mathbb{Q}}_\ell)_\phi &\cong R\Gamma_c(Z^H_{\phi,r},\overline{\mathbb{Q}}_\ell)_\phi\otimes R\Gamma_c(Z^L_{\phi,r},\overline{\mathbb{Q}}_\ell)_\phi \\ &\cong \left(R\Gamma_c(Z^H_{\phi,r},\overline{\mathbb{Q}}_\ell)_\phi\otimes\prod_{i=0}^{d}\phi_i\right)\otimes R\Gamma_c(Z^L_{\phi,0},\overline{\mathbb{Q}}_\ell)_{\phi_{-1}} \\ &\cong \pm\kappa_\phi\otimes R\Gamma_c(Z^L_{\phi,0},\overline{\mathbb{Q}}_\ell)_{\phi_{-1}}. \end{aligned}$$

The coincidence

Proposition (Chan, N.)

Assume T is elliptic and $\phi \in \widehat{T^F}$ is of depth $\leqslant r$. Then

$$\langle R_{T_r}^{G_r}(\phi), R_{T_r}^{G_r}(\phi)\rangle_{G_r^F} = \langle R_{T_r}^{G_r}(\phi), \mathcal{R}_{T_r}^{G_r}(\phi)\rangle_{G_r^F} = \langle \mathcal{R}_{T_r}^{G_r}(\phi), \mathcal{R}_{T_r}^{G_r}(\phi)\rangle_{G_r^F}.$$

In particular, $R_{T_r}^{G_r}(\phi) = \mathcal{R}_{T_r}^{G_r}(\phi)$.

Remark

When ϕ is generic and $\mathbf{x}$ is hyperspecial, the coincidence is proved Chen-Stasinski.

A decomposition result

Theorem (N.)

Assume T is elliptic. There is an irreducible decomposition

$$R_{T_r}^{G_r}(\phi) = \mathcal{R}_{T_r}^{G_r}(\phi) \cong \mathrm{ind}_{K_{\phi,r}^F}^{G_r^F} \kappa_\phi \otimes R_{T_0}^{L_0}(\phi_{-1}) = \sum_\rho m_\rho \mathrm{ind}_{K_{\phi,r}^F}^{G_r^F} \kappa_\phi \otimes \rho,$$

where $\mathrm{ind}_{K_{\phi,r}^F}^{G_r^F} \kappa_\phi \otimes \rho$ are pairwise non-isomorphic irreducible rep'ns.

Corollary (Chan-Oi, N.)

Each irre. supercuspidal rep'n of G^F is a direct summand of

$$c\text{-}\mathrm{ind}_{Z^F G_{x,0}^F}^{G^F} H_c^i(Y_r, \overline{\mathbb{Q}}_\ell)_\phi$$

for some elliptic T, ϕ and i.

Example: Kaletha's regular supercuspidal rep'ns

Proposition (Chan-Oi, N.)

If T is elliptic and ϕ is regular in the sense of Kaletha, then

$$\pi_{T,\phi} := \pm c\text{-}\mathrm{ind}_{Z^F G^F_{x,0}}^{G^F} R_{T_r}^{G_r}(\phi)$$

is an irreducible supercuspidal rep'n of G^F.

Remark

Chan-Oi noticed that the map $(T, \phi) \mapsto \pi_{T,\phi}$ coincides with the LLC for Kaletha's regular supercuspidal rep'ns.

A concentration result

Theorem (Ivanov-N.)

There exists $B \supseteq T$ and $m \in \mathbb{Z}$ such that

$$R\Gamma_c(Y_r, \overline{\mathbb{Q}})_\phi \cong R\Gamma_c(Z_{\phi,r})_\phi[m].$$

In particular, if ϕ_{-1} is non-singular for $L = G^0$, the complex $R\Gamma_c(Y_r, \overline{\mathbb{Q}})_\phi$ concentrates at a single cohomological degree.

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In particular, if ϕ_{-1} *is non-singular for* $L = G^0$*, the complex* $R\Gamma_c(Y_r, \overline{\mathbb{Q}})_\phi$ *concentrates at a single cohomological degree.*

Remark

It extends results of Boyarchenko-Weinstein and Chan-Ivanov in the case of inner forms of GL_n.

Alternative construction of Character sheaves

Let $\phi \in \widehat{T}^F$ of depth $r \geqslant 0$.

- Consider $\widetilde{G}_r = \{(g, hI_{\phi,r}) \in G_r \times G_r/I_{\phi,r}; h^{-1}gh \in I_{\phi,r}\}$.
- There are natural maps

$$T_r/T_{\phi,r} \xleftarrow{f} \widetilde{G}_r \xrightarrow{\pi} G_r, \qquad \mathrm{pr}_{T_r}(h^{-1}gh) \leftarrow (g, hI_{\phi,r}) \to g.$$

- define

$$\mathrm{Ind}_{I_{\phi,r}}^{G_r} : D_{T_r}(T_r/T_{\phi,r}) \to D_{G_r}(G_r), \qquad \mathcal{F} \mapsto \pi_! f^* \mathcal{F}[2\dim G_r/I_{\phi,r}].$$

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$$T_r/T_{\phi,r} \xleftarrow{f} \widetilde{G}_r \xrightarrow{\pi} G_r, \qquad \mathrm{pr}_{T_r}(h^{-1}gh) \leftarrow (g, hI_{\phi,r}) \to g.$$

- define

$$\mathrm{Ind}_{I_{\phi,r}}^{G_r} : D_{T_r}(T_r/T_{\phi,r}) \to D_{G_r}(G_r), \qquad \mathcal{F} \mapsto \pi_! f^* \mathcal{F}[2\dim G_r/I_{\phi,r}].$$

Remark

If ϕ is generic, this construction is first introduced by Lusztig.

Alternative parabolic induction/restriction functors

Let $T \subseteq P = LN \subseteq G$ be parabolic subgroup. Take $s = r/2$ and define

$$P_{s,r} = L_r N_{s:r} \overline{N}_{s+:r}.$$

Consider the natural correspondence of stacks

$$\begin{array}{ccc} & P_{s,r}/P_{s,r} & \\ \overset{\alpha}{\swarrow} & & \overset{\beta}{\searrow} \\ L_r/L_r & & G_r/G_r. \end{array}$$

The induction/restriction functors are defined by

$$\mathrm{Ind}_{P_{s,r}}^{G_r} = \beta_! \circ \alpha^* : D_{L_r}(L_r) \longrightarrow D_{G_r}(G_r);$$
$$\mathrm{Res}_{P_{s,r}}^{G_r} = \alpha_* \circ \beta^! : D_{G_r}(G_r) \longrightarrow D_{L_r}(L_r).$$

t-exactness

Theorem (Ivanov-N.-Yu)

The parabolic induction restricts to a t-exact equivalence

$$\mathrm{Ind}_{P_{s,r}}^{G_r} : D_{L_r}^{\psi}(L_r) \cong D_{G_r}^{\psi}(G_r),$$

whose inverse is $\mathcal{L}_{\psi,r} \star_* \mathrm{Res}_{P_{s,r}}^{G_r}$.

Remark

The proof follows the strategy of Bezrukavonikov-Chan.

Iterated formulation

Recall the Howe factorization $(G^i, \phi_{-1})_{-1 \leqslant i \leqslant d}$ of ϕ. Consider

$$\Psi_i = \mathcal{L}_{\phi_i} \otimes \varepsilon_i^* \mathrm{Ind}_{P^i_{s_{i-1}, r_{i-1}}}^{G^i_{r_{i-1}}} [\dim G^i_{r_{i-1}} - \dim G^i_{r_i}] : D_{G^{i-1}_{r_{i-1}}}(G^{i-1}_{r_{i-1}}) \to G_{G^i_{r_i}}(G^i_{r_i}).$$

Here $G^{i-1} \subseteq P^i \subseteq G^i$ is a parabolic subgroup and $\varepsilon_i : G^i_{r_i} \to G^i_{r_{i-1}}$ is the projection.

Proposition (Ivanov-N.-Yu)

We have

$$\mathrm{Ind}_{I_{\phi,r}}^{G_r}(\mathcal{L}_\phi[N_\phi]) = \Psi_d \circ \cdots \circ \Psi_0(\mathcal{L}_{\phi_{-1}}[\dim T_0]).$$

Here $N_\phi = \dim T_0 + \sum_{i=0}^d (\dim G^i_{r_i} - \dim G^i_{r_{i-1}})$.

Main results

Theorem (Ivanov-N.-Yu)

We have that

$$\mathrm{Ind}_{I_{\phi,r}}^{G_r}(\mathcal{L}_\phi[N_\phi]) \cong j_{!*}(\pi_{\mathrm{vreg}!} f_{\mathrm{vreg}}^{*} \mathcal{L}_{\mathrm{vreg}}[\dim G_r])$$

is a semisimple perverse sheaf.

If ϕ is regular, then

$$\chi_{\mathrm{Ind}_{I_{\phi,r}}^{G_r}(\mathcal{L}_\phi[N_\phi])} = (-1)^{\dim G_r} R_{T_r}^{G_r}(\phi).$$

Corollary

$\mathrm{Ind}_{I_{\phi,r}}^{G_r}(\mathcal{L}_\phi[N_\phi])$ *coincides with the iterated construction of Bezrukavnikov-Chan using the functors* $\mathrm{Ind}_{P_{r_{i-1}}}^{G^i_{r_{i-1}}}$.

Deep level Springer hypothesis

Let $\mathbf{g} = \mathrm{Lie}(G)$ and $\psi : k \to \overline{\mathbb{Q}}_\ell^\times$ an additive character of level 1.

- put $\mathbf{g}_r = \mathbf{g}_{\mathbf{x},0}/\mathbf{g}_{\mathbf{x},r+}$ and $\mathbf{g}^*_{-r} = \mathbf{g}^*_{\mathbf{x},-r}/\mathbf{g}_{\mathbf{x},0+}$.
- let $T_\psi : C((\mathbf{g}^*_{-r})^F) \to C(\mathbf{g}_r^F)$ be the Fourier transform of functions

$$T_\psi(f)(Y) := \sum_{Z \in (\mathbf{g}^*_{-r})^F} f(Z)\psi(\langle Z, Y\rangle).$$

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Theorem (Chan-Oi, Ivanov-N.-Yu)

Let $\phi \in \widehat{T^F}$ be of depth r. Then $\exists$ a regular element $X \in (\mathbf{g}^*_{-r})^F$ such that

$$R_{T_r}^{G_r}(u) = q^{-\dim M_\phi} T_\psi(1_{G_r^F . X})(\log(u)) \quad \textit{for} \quad u \in (G_r)_{\mathrm{unip}}.$$

Here $M_\phi = \sum_{i=0}^d \dim(G^i_{r_{i-1}}/G^{i-1}_{r_{i-1}})$.

Further question

Suppose $T \subseteq G$ is a tamely ramified, maximally unramified, elliptic maximal torus.

Theorem (Chan-Kaletha-Zhu, Ivanov-N.)

The construction of $Z_{\phi,r}$ can be generalized by taking Lagrangian subspaces of H_ϕ/K_ϕ^+. Moreover,

$$H_c^*(Z_r, \overline{\mathbb{Q}}_\ell)_\phi \cong \pm \mathrm{ind}_{K_{\phi,r}^F}^{G_r^F} \kappa \otimes R_{T_0}^{L_0}(\phi_{-1}^\dagger).$$

Question

Is $\mathrm{Ind}_{I_{\phi,r}}^{G_r} \mathcal{L}_\phi[N_\phi]$ still a perverse sheaf?

Thanks!