

On equidistribution and Howe duality for finite fields

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Outline

1 Equidistribution

2 Howe duality

Commutators in finite simple groups

- **History: (Ore conjecture)** For a finite (non-abelian) simple group G , is it true that every element $g \in G$ can be expressed as a commutator

$$g = xyx^{-1}y^{-1}?$$

Theorem (Liebeck, O'Brien, Shalev, Tiep)

The Ore conjecture holds.

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Equidistribution

- **Shalev conjecture:** For a family of finite simple groups, e.g. finite groups of Lie type, do the commutators (excluding 1) become equidistributed?
- **More specifically:** Fix a finite reductive group $G(\mathbb{F}_q)$, and denote the commutator map as

$$\begin{aligned} c_{G(\mathbb{F}_q)} : G(\mathbb{F}_q) \times G(\mathbb{F}_q) &\rightarrow G(\mathbb{F}_q) \\ (x, y) &\mapsto xyx^{-1}y^{-1} \end{aligned}$$

Is it true that for each $g \in G(\mathbb{F}_{q_0}) \setminus Z(G(\mathbb{F}_{q_0}))$,

$$\lim_{q \rightarrow \infty} \frac{|c_{G(\mathbb{F}_q)}^{-1}(g)|}{|G(\mathbb{F}_q)|} = 1?$$

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■ **Liebeck, Shalev** (with recent corrections by **Deligne**):

- For $G = \mathrm{PSL}_3$, g a transvection,

$$\frac{|c_{\mathrm{PSL}_3(\mathbb{F}_q)}^{-1}(g)|}{|\mathrm{PSL}_3(\mathbb{F}_q)|} = \begin{cases} 2 + O(q^{-1}) & \text{if } q \equiv 1 \pmod{3} \\ 4/3 + O(q^{-1}) & \text{if } q \equiv -1 \pmod{3} \end{cases}$$

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$$\frac{|c_{\mathrm{SL}_n(\mathbb{F}_q)}^{-1}(g)|}{|\mathrm{SL}_n(\mathbb{F}_q)|} \geq K \cdot \sqrt{n}$$

- P. Deligne remarked that this implies that the fiber

$$c_{\mathrm{SL}_n(\overline{\mathbb{F}_q})}^{-1}(g)$$

has many irreducible components. In a recent manuscript, Larsen and Tiep proved that in fact it has $n - 1$ irreducible components of dimension $n^2 - 1$.

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The case of Sp_{2N}

Theorem (S.K.)

For every $g \in Sp_{2N}(\mathbb{F}_{q_0}) \setminus \{\pm I\}$,

$$\lim_{q \rightarrow \infty} \frac{|c_{Sp_{2N}(\mathbb{F}_q)}^{-1}(g)|}{|Sp_{2N}(\mathbb{F}_q)|} = 1.$$

(we consider this limit to be over a sequence of prime powers $q_0^{n_i}$ such that $n_i | n_{i+1}$ and every natural number divides an eventual n_j).

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Some corollaries - Uniformity

Theorem (P. Deligne)

*The number of pairs $(x, y) \in \mathrm{Sp}_{2N}(\mathbb{F}_q) \times \mathrm{Sp}_{2N}(\mathbb{F}_q)$ such that $xyx^{-1}y^{-1} = g$, divided by the group order $|\mathrm{Sp}_{2N}(\mathbb{F}_q)|$, converges to 1 as q tends to ∞ as a **natural number**, uniformly in all odd prime powers q and all $g \in \mathrm{Sp}_{2N}(\mathbb{F}_q) \setminus Z(\mathrm{Sp}_{2N}(\mathbb{F}_q))$.*

Proof sketch:

- Consider the commutator map

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has exactly 1 irreducible component of dimension $\dim(\mathrm{Sp}_{2N}) = 2N^2 + N$ (and no irreducible components of higher dimension).

- The constructibility of $R^i c_! \underline{\mathbb{Q}}_\ell$ on Sp_{2N} over $\mathrm{Spec}(\mathbb{Z})$ then gives the uniform limit statement.



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On the proof of equidistribution for Sp_{2N}

- **Character ratio sums:** It is a classical result of Frobenius that

$$|c_G^{-1}(g)| = |G| \cdot \sum_{\rho \in \widehat{G}} \frac{\chi_{\rho}(g)}{\dim(\rho)}$$

- In particular, to prove equidistribution, it suffices to show

$$\lim_{q \rightarrow \infty} \sum_{\rho \neq 1 \in \widehat{\mathrm{Sp}_{2N}(\mathbb{F}_q)}} \frac{\chi_{\rho}(g)}{\dim(\rho)} = 0.$$

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A very brief review of Deligne-Lusztig theory

- Consider a finite reductive group G and a parabolic $P = LU$ (over $\overline{\mathbb{F}}_q$) with $\mathbb{F}_q$ -rational Levi factor L and unipotent radical U . Consider the Lang map

$$\begin{aligned}\mathcal{L} : G &\rightarrow G. \\ x &\mapsto F(x)^{-1} \cdot x\end{aligned}$$

- Consider $\mathcal{L}^{-1}(U)$. This is (a version of) the **Deligne-Lusztig variety**, with $G(\mathbb{F}_q)$ acting on the left and $L(\mathbb{F}_q)$ acting on the right.

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- For a given $L(\mathbb{F}_q)$ -representation ρ , we may form the virtual $G(\mathbb{F}_q)$ -character

$$R_G^L(\rho) = \sum_i (-1)^i H_c^i(\mathcal{L}^{-1}(U))[\rho]$$

- Considering specifically the case of tori $L = T$, every irreducible $G(\mathbb{F}_q)$ -representation appears with non-zero multiplicity in some Deligne-Lusztig induction $R_G^T(\theta)$ for a character θ .

- Consider the dual group $G^*(\mathbb{F}_q)$ so that the data

(torus T in G , character θ of $T(\mathbb{F}_q)$)

corresponds to data

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- For each $(s) \in G^*(\mathbb{F}_q)$, we then consider the set

$$\mathcal{E}(G(\mathbb{F}_q), (s)) \subset \widehat{G(\mathbb{F}_q)}$$

of irreducible representations ρ such that there exists (T, θ) corresponding to (s) where ρ appears with non-zero coefficient in $R_T^G(\theta)$.

- The representations in $\mathcal{E}(G(\mathbb{F}_q), 1)$ are called the irreducible *unipotent* representations and are indexed by combinatorial data determined by the Weyl group of G , so that $\mathcal{E}(G(\mathbb{F}_q), 1)$ can be identified with $\mathcal{E}(G^{ad}(\mathbb{F}_q), 1)$.

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- **Example:** For $\mathrm{Sp}_{2N}(\mathbb{F}_q)$, the irreducible unipotent representations are indexed by *symbols* of the form

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

such that $a - b$ is odd and $\sum \lambda_i + \sum \mu_j = N + \frac{(a+b-1)^2}{4}$, $(\lambda_1, \mu_1) \neq (0, 0)$.

- **Lusztig classification theorem:** For a group G with connected center, these sets form a disjoint partition

$$\widehat{G(\mathbb{F}_q)} = \coprod \mathcal{E}(G(\mathbb{F}_q), (s)),$$

and satisfy a bijective correspondence

$$\mathcal{E}(G(\mathbb{F}_q), (s)) \xrightarrow{\cong} \mathcal{E}(Z_{G^*}(\mathbb{F}_q)(s)^*, 1)$$

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- $Z(\mathrm{Sp}_{2N}(\mathbb{F}_q)) = \mathbb{Z}/2$. For G with disconnected center, we may still consider, for an irreducible character θ of a torus $T(\mathbb{F}_q) \subseteq G(\mathbb{F}_q)$ (up to $G^{ad}(\mathbb{F}_q)$ -conjugation), the representation

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- This representation may then further split into representations conjugate under the action $G^{ad}(\mathbb{F}_q)$ which are non-isomorphic over $G(\mathbb{F}_q)$ (specifically, in the case when the $G^{ad}(\mathbb{F}_q)$ -conjugacy class of θ is larger than the $G(\mathbb{F}_q)$ -conjugacy class)

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G. Lusztig's parametrization of characters

The representations of a finite symplectic group are classified by the following data:

- **Semisimple data:** A semisimple conjugacy class (s) in the dual group $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$.
- **Unipotent data:** A unipotent representation u of (the dual of) the centralizer $Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(s)$.
- **Sign data:** In certain cases, (if s has -1 eigenvalues and the corresponding symbol factor of u is non-degenerate), we also fix the data of a sign ± 1 .

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Standard example (the Drinfeld curve): $\mathrm{SL}_2(\mathbb{F}_q)$, with dual group is $\mathrm{SO}_3(\mathbb{F}_q)$.

- Taking $(s) = I$, there are two **unipotent** representations

$$1 = \begin{pmatrix} 1 \\ \emptyset \end{pmatrix}, \quad \mathrm{St} = \begin{pmatrix} 0 < 1 \\ 1 \end{pmatrix}$$

- Taking (s) with -1 eigenvalues gives **oscillator** representations:

$$(s) = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix},$$

$$Z(s)^\circ = \mathrm{so}_2^+(\mathbb{F}_q) \leadsto u = 1$$

The two sign data give

$$\boxed{\omega_a^+, \omega_b^+}.$$

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with $Z(S) = \mathrm{GL}_1(\mathbb{F}_q) \leadsto u = 1$, gives an irreducible representation of dimension $q + 1$. This is the **principal series**.

- For each $\{\mu, \mu^{-1}\} \subseteq \mathbb{F}_{q^2}^\times \setminus \{\pm 1\}$ with $N(\mu) = 1$, writing $\mu = x + y\sqrt{\alpha}$,

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The method of proving equidistribution for Sp_{2N}

- Define a representation $\rho \in \widehat{\mathrm{Sp}_{2N}(\mathbb{F}_q)}$ to be *primitive* if the only eigenvalues of the semisimple data corresponding to ρ are ± 1 .
- We split the character ratio sum as

$$\underbrace{\sum_{\substack{\rho \neq 1 \in \widehat{\mathrm{Sp}_{2N}(\mathbb{F}_q)} \\ \text{primitive}}} \frac{\chi_\rho(g)}{\dim(\rho)}}_{\substack{\text{number of summands} \\ \text{does not depend on } q}} + \underbrace{\sum_{\substack{\rho \in \widehat{\mathrm{Sp}_{2N}(\mathbb{F}_q)} \\ \text{non-primitive}}} \frac{\chi_\rho(g)}{\dim(\rho)}}_{\substack{\text{cancellation among} \\ \text{summands}}}$$

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The method of proving equidistribution for Sp_{2N}

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- Consider a semisimple element $(s) \in G^*(\mathbb{F}_q)$ such that there exists a Levi subgroup $L^* \subseteq G^*$ containing s 's centralizer.
- Then for every unipotent irreducible representation u of $Z_{G^*}(s) = Z_{L^*}(s)$, the $L(\mathbb{F}_q)$ -representation $\rho_L[(s), u]$ and the $G(\mathbb{F}_q)$ -representation $\rho_G[(s), u]$ are related by Deligne-Lusztig induction

$$R_G^L(\rho_L[(s), u]) = \rho_G[(s), u].$$

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- Consider for $g \neq \pm I$

$$\deg_q \left(\sum_{\lambda \in \mu_{q^r-1}^{\text{gen}}} \frac{\chi_{R_{\text{Sp}_{2N}}^{L+}(\rho' \otimes u_\lambda)}(g)}{\dim(R_{\text{Sp}_{2N}}^{L+}(\rho' \otimes u_\lambda))} + \sum_{\lambda \in \mu_{q^r+1}^{\text{gen}}} \frac{\chi_{|R_{\text{Sp}_{2N}}^{L-}(\rho' \otimes u_\lambda)|}(g)}{\dim(|R_{\text{Sp}_{2N}}^{L-}(\rho' \otimes u_\lambda)|)} \right).$$

- Using recursion, we prove it is

$$< 2(nr - N) \quad \text{or} \quad < \deg_q \left(\frac{\chi_{\rho'}(g')}{\dim(\rho')} \right)$$

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Recursion

- The character (and dimension) of a Deligne-Lusztig induction can be calculated geometrically in terms of the character values of the input $\rho' \otimes u_\lambda$.
- The key example is in the case of the advection g , $n = 1$:
For $N - N' = 1$ (corresponding to $L^\pm = \mathrm{Sp}_{2(N-1)}(\mathbb{F}_q) \times \mathrm{U}_1^\pm(\mathbb{F}_q)$),

$$\frac{\chi_{R_{\mathrm{Sp}_{2N}}^{L^\pm}(\rho' \otimes \psi)}(g)}{\dim(R_{\mathrm{Sp}_{2N}}^{L^\pm}(\rho' \otimes \psi))} = \pm \frac{q \mp 1}{q^{2N} - 1} \pm q \frac{q^{2(N-1)} - 1}{q^{2N} - 1} \cdot \frac{\chi_{\rho'}(g')}{\dim(\rho')},$$

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- For g not an advection or $L^\pm$ with a factor $U_n^\pm(\mathbb{F}_{q^r})$, $n > 1$,

$$\deg_q \left(\sum_{\lambda \in \mu_{q^r \mp 1}^{\text{gen}}} \frac{\chi_{R_{\text{Sp}_{2N}}^{L^\pm}(\rho' \otimes u_\lambda)}(g)}{\dim(R_{\text{Sp}_{2N}}^{L^\pm}(\rho' \otimes u_\lambda))} \right) < 0$$

since we have

$$\text{mult}_1^{\text{geom}}(g) < 2N - 1$$

or for each summand, $\lambda \in \mu_{q^r \mp 1}^{\text{gen}}$ appears as an eigenvalue of multiplicity > 1 in a semisimple element with centralizer L^+ , so that the number of summands is overpowered by the dimension in the denominator as $q \rightarrow \infty$.

Bounding individual character ratios

- To prove equidistribution, it remains to show, for $g \neq \pm I \in \mathrm{Sp}_{2N}(\mathbb{F}_q)$

$$\lim_{q \rightarrow \infty} \sum_{\substack{\rho \neq 1 \in \widehat{\mathrm{Sp}_{2N}(\mathbb{F}_q)} \\ \text{primitive}}} \frac{\chi_\rho(g)}{\dim(\rho)} = 0$$

In fact, suffices to show each $\chi_\rho(g)/\dim(\rho)$ vanishes.

- Extreme cases:

$$g \in \mathrm{GL}_N(\mathbb{F}_q) \subseteq \mathrm{Sp}_{2N}(\mathbb{F}_q), \quad g = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$$

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- **Step 1:** For every $\rho \neq \psi(\det) \in \widehat{\mathrm{GL}}_N(\mathbb{F}_q)$ the character ratio $\chi_\rho(g)/\dim(\rho)$ is $o(1)$. ($\leadsto$ Similarly, for $\rho \in \widehat{P}_N$.)
- **Step 2:** Fix an irreducible $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ -representation ρ .
 - **Case A:** There are no $\psi(\det)$ in $\mathrm{Res}_{P_N}(\rho)$.
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Fix $\rho \neq 1 \in \widehat{\mathrm{Sp}_{2N}(\mathbb{F}_q)}$.

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$$g = \bigoplus_{i=1}^k \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix} \oplus I_{2(N-k)} \in \prod_{i=1}^k \mathrm{SL}_2(\mathbb{F}_q) \times \mathrm{Sp}_{2(N-k)}(\mathbb{F}_q)$$

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In-between cases

- **Step 1:** Prove $\chi(\rho)/\dim(\rho)$ is $o(1)$ for irreducible representations ρ of the maximal parabolic $P_N = \mathrm{GL}_N(\mathbb{F}_q) \ltimes \mathrm{Sym}^2(\mathbb{F}_q^N)$ first.
 - Have $g = g_{\mathrm{GL}_N(\mathbb{F}_q)} \cdot g_{\mathrm{Sym}^2(\mathbb{F}_q^N)}$.
 - Follows if restriction of ρ to $\mathrm{Sym}^2(\mathbb{F}_q^N)$ is trivial.
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Outline

1 Equidistribution

2 Howe duality

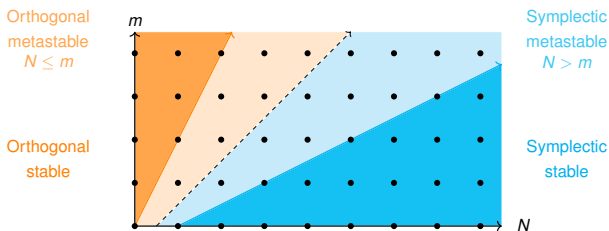
Howe duality problem (for Type I dual pairs)

- How does the oscillator $\mathrm{Sp}(V \otimes W)$ -representation $\omega[V \otimes W]$ decompose under restriction to $\mathrm{Sp}(V) \times \mathrm{O}(W, B)$ (over a finite field $\mathbb{F}_q$)?
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- I developed a new direct method using the study of endomorphism algebras and interpolation theory, completely describing the pairs which occur explicitly by breaking up all the cases into four ranges:



The method

- The $\star$ product on $\mathbb{C}V$:

$$(v) \star (w) = \psi\left(\frac{1}{2}S(v, w)\right) \cdot (v + w).$$

- Action on $\omega = \mathbb{C}V_-$: For $x \in V_-$,

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$$(v)[(x)] = \psi\left(S(v_+, x) + \frac{1}{2}S(v_+, v_-)\right) \cdot (v_- + x)$$

- “Standard” Heisenberg group product:

$$(v) * (w) = \psi(S(v_+, w_-)) \cdot (v + w)$$

- There is an isomorphism

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- Corollary: **Howe's Theorem**

$$\omega \otimes \omega^\vee \cong \mathbb{C}V$$

- This identifies $\text{End}_{\text{Sp}(V)}(\omega[V \otimes W])$ with

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Interpolation can be used to obtain the answer in the metastable ranges. There are two ways in which the stable situation decays:

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- Can the oscillator representation be used to bound the character ratio sum? Two effects were observed computationally:
 - 1 Representations ρ of “small size” contribute the largest character ratios
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- More specifically, they defined **stable eta correspondences**

$$\eta_{n,\pm}^{2N} : \widehat{O_n(\mathbb{F}_q, \pm)} \rightarrow \widehat{Sp_{2N}(\mathbb{F}_q)}$$

for $n \leq N$, which send an $O_n(\mathbb{F}_q, \pm)$ -representation π to the “top” $Sp_{2N}(\mathbb{F}_q)$ -representation ρ such that

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The explicit computation can be used to prove all “small” representations are produced in the images of stable eta correspondences.

The concrete notion of “size” Gurevich and Howe use is **U -rank**, defined for a $\rho \in \widehat{\mathrm{Sp}_{2N}(\mathbb{F}_q)}$ as the maximal rank of a symmetric matrix corresponding to a character in the restriction of ρ to the Siegel unipotent U (by the standard identification of U with its Pontrjagin dual).

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Thank you!