Enhanced Langlands parameters and Hecke algebras

Anne-Marie Aubert

Institut de Mathématiques de Jussieu – Paris Rive Gauche
C.N.R.S., Sorbonne Université and Université Paris Cité

MIT Lie Groups Seminar
20 March 2024
Introduction

Notation

- $G$: group of $F$-rational points of a connected reductive algebraic $F$-group, with $F$ a non-archimedean local field (finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$). We will refer to $G$ as a $p$-adic group.
- $W_F$: Weil group of $F$
- $G^\vee$: complex reductive group with root datum dual to that of $G$
- $L^G := G^\vee \rtimes W_F$: the $L$-group of $G$

The Local Langlands Correspondence (LLC)

predicts a surjective map, satisfying several properties,

\[
\left\{ \text{irred. smooth repres. } \pi \text{ of } G \right\} \xrightarrow{\text{iso.}} \left\{ \text{i.e. cont. homomorphisms } \varphi_\pi: W_F \times \text{SL}_2(\mathbb{C}) \to L^G \right\} / G^\vee\text{-conj.}
\]

with finite fibers, called $L$-packets.
In order to obtain a bijection LLC between the group side and the Galois side, the conjectural map $\mathcal{L}$ was later enhanced: on the Galois side, one considers enhanced $L$-parameters: $(\varphi_\pi, \rho_\pi)$, where the enhancement $\rho_\pi$ is a representation of a certain component group.

It may be useful to consider simultaneously inner twists of a given group $G$. This leads to “compound” $L$-packets.

There are several ways one can enhance $L$-parameters in order to capture information about the internal structure of $L$-packets. For the most part, the choice of enhancement has to do with the type of inner twist of $G$ we consider.
A bijective LLC

has been constructed in particular in the following cases:

- $G = F^\times = \text{GL}_1(F)$ Class field theory (first half of the 20th century);
- $G = \text{GL}_n(F)$ Laumon-Rapoport-Stuhler (1993) $\text{char}(F) > 0$, Harris-Taylor (1998), Henniart (2000), Scholze (2010);
- $G = \text{SL}_n(F)$ (and its inner twists) Hiraga-Saito (2012) $\text{char}(F) = 0$; A.-Baum-Plymen-Solleveld (2016) $\text{char}(F) > 0$;
- $G = \text{Sp}_{2n}(F), \text{SO}_{2n+1}(F)$ (char$(F) = 0$) Arthur (2013);
- $G = G_2(F)$ A.-Xu (2022), Gan-Savin (2022);
The Bernstein decomposition [Bernstein, 1984]

The category $\mathcal{R}(G)$ of smooth representations of a $p$-adic group $G$ is a direct product

$$\mathcal{R}(G) = \prod_{s \in \mathcal{B}(G)} \mathcal{R}^s(G) \quad (1)$$

of the full subcategories $\mathcal{R}^s(G)$, where

- $\mathcal{B}(G) = \{s = (L, X_{nr}(L) \cdot \sigma)_G\}$. Notation $s = [L, \sigma]_G$.
  - $L$ Levi subgroup of $G$ and $\sigma$ supercuspidal smooth irrep of $L$
  - $X_{nr}(L)$ group of unramified characters of $L$

- $\mathcal{R}^s(G)$ subcategory of $\mathcal{R}(G)$ whose objects are the representations $\pi$ such that every irreducible $G$-subquotient of $\pi$ has its supercuspidal support in $s$.

Example: The irred. objects of $\mathcal{R}^{s_1}(G)$, where $s_1 = [T, \text{triv}]_G$, are the Iwahori-spherical irreps. of $G$. 
An extended finite Weyl group

Set $W_s := N_G(s)/L$. It is an extended finite Weyl group:

$$W_s = W_s^\circ \rtimes \Gamma_s$$

where $W_s^\circ$ is the finite Weyl group, with root system $\Sigma_s$, the set of roots for which the associated Harish-Chandra $\mu$-function has a zero on $X_{nr}(L) \cdot \sigma$, and $\Gamma_s$ is the stabilizer of the set of positive roots.

A root datum attached to $s$

Set $X_{nr}(L, \sigma) := \{\chi \in X_{nr}(L) : \sigma \otimes \chi \cong \sigma\}$ and $L_\sigma := \bigcap_{\chi \in X_{nr}(L, \sigma)} \ker \chi$. Let $L_1$ be the subgroup of $L$ generated by all compact subgroups of $L$.

Let $\alpha \in \Sigma_s$ and $h_\alpha^\vee$ the unique generator of $\left( L_\sigma \cap L_1 \right)/L_1 \cong \mathbb{Z}$ such that $|\alpha(h_\alpha^\vee)|_F > 1$. We write $R_s := \{h_\alpha^\vee : \alpha \in \Sigma_s\}$.

Then $T_s := X_{nr}(L)/X_{nr}(L, \sigma)$ is a complex torus and

$$\mathcal{R}_s := (X^*(T_s), R_s, X^*_s(T_s), R^\vee_s)$$

is a root datum.
Weights functions

These are functions $\lambda, \lambda^*: R_5 \to \mathbb{R}_{\geq 0}$, such that

- if $\alpha, \beta \in R_5$ are $W_5^o$-associate, then $\lambda(\alpha) = \lambda(\beta)$ and $\lambda^*(\alpha) = \lambda^*(\beta)$,
- if $\alpha^\vee \notin 2X_*(T_5)$, then $\lambda^*(\alpha) = \lambda(\alpha)$. (It is always the case except possibly for short roots $\alpha$ in a type B component of $R_5$.)

They are defined by

$$
\lambda(h^\vee_\alpha) := \log(q_\alpha q_{\alpha^*}) / \log(q) \quad \text{and} \quad \lambda^*(h^\vee_\alpha) := \log(q_\alpha q_{\alpha^*}^{-1}) / \log(q_\alpha),
$$

where $q_\alpha, q_{\alpha^*} \in \mathbb{R}_{\geq 1}$ come from Silberger’s computation of the Harish-Chandra $\mu$-function associated to $\alpha$. 
The affine Hecke algebra $\mathcal{H}(\mathcal{R}_s, \lambda, \lambda^* q^{1/2})$ is the vector space $\mathcal{H}(\mathcal{W}_s^\circ, q^{\lambda(\alpha)}) \otimes_{\mathbb{C}} \mathbb{C}[X^*(T_s)]$ with the multiplication rules:

- $\mathcal{H}(\mathcal{W}_s^\circ, q^{\lambda(\alpha)})$ and $\mathbb{C}[X^*(T_s)]$ are embedded as subalgebras;
- for $\alpha \in \Delta_s$ (a basis of $R_s$) and $x \in X^*(T_s)$:

$$\theta_x T_{s\alpha} - T_{s\alpha} \theta_{s\alpha}(x) =$$

$$\left( (q^{\lambda(\alpha)} - 1) + \theta_{-\alpha}(q^{(\lambda(\alpha) + \lambda^*(\alpha))/2} - q^{(\lambda(\alpha) - \lambda^*(\alpha))/2}) \right) \frac{\theta_x - \theta_{s\alpha}(x)}{\theta_0 - \theta_{-2\alpha}},$$

where $\{\theta_x : x \in X\}$ is a basis of $\mathbb{C}[X^*(T_s)]$.

It is an associative algebra with unit element $T_1 \otimes \theta_0$. 
Structure of blocks [Heiermann, Solleveld]

In many cases, it is known that

$$\mathcal{R}^s(G) \cong \text{Mod}(\mathcal{H}(G, s))$$

(3)

where \(\mathcal{H}(G, s)\) is a (twisted) extended affine Hecke algebra:

$$\mathcal{H}(G, s) = \mathcal{H}(G, s)^\circ \rtimes \mathbb{C}[\Gamma_s, \Lambda_s],$$

(4)

and \(\mathcal{H}(G, s)^\circ = \mathcal{H}(\mathcal{R}_s, \lambda, \lambda^* q^{1/2})\).

Remark

For instance (3) is satisfied if the restriction of \(\sigma\) to \(L_1\) is multiplicity free. It is the case, in particular, when the maximal \(F\)-split central torus of \(L\) has dimension \(\leq 1\), and also when \(L\) is quasi-split and \(\sigma\) is generic.
A general strategy to construct the LLC (A.-Moussaoui-Solleveld):

1. Define an analogue of Bernstein’s decomposition on the Galois side of the correspondence.

2. Attach a (twisted) extended affine Hecke algebra to each “Galois block”.

3. Construct an explicit LLC for $p$-adic groups “block by block” via a correspondence between Hecke algebras: prove that the (twisted) extended affine Hecke algebras on each side of the correspondence are isomorphic, or at least closely related: in particular, we need that their simple modules are in bijection.

Theorem [A.-Moussaoui-Solleveld, 2023]

The above strategy works for all pure inner forms of quasi-split $p$-adic classical groups (symplectic, (special) orthogonal, general (s)pin, and unitary groups) and the obtained correspondence coincides with Arthur’s LLC.
Definition

For simplicity, suppose $G$ pure inner twist of a quasi-split group. We set

$$S_{\varphi} := Z_G^\vee(\varphi(W'_F)).$$  \(5\)

An enhanced $L$-parameter is a pair $(\varphi, \rho)$ where $\varphi$ is an $L$-parameter for $G$ and $\rho \in \text{Irr}(S_{\varphi})$, with $S_{\varphi} := S_{\varphi}/S_\circ$.

For $\varphi$ a given $L$-parameter, $\rho$ is called an enhancement of $\varphi$.

Action of $G^\vee$ on the set of enhanced $L$-parameters:

$$g \cdot (\varphi, \rho) := (g\varphi g^{-1}, g^\rho), \text{ for } g \in G^\vee,$$

where $g^\rho : h \mapsto \rho(g^{-1}hg)$.

$\Phi_e$: set of $G^\vee$-conjugacy classes of enhanced $L$-parameters.

$\Phi_e(G)$: set of $G^\vee$-conjugacy classes of $G$-relevant enhanced $L$-parameters.
The role of the generalized Springer correspondences

Definitions

- \( G_\varphi := Z_{G^\vee}(\varphi(W_F)) \): a (possibly disconnected) complex reductive group
- \( u = u_\varphi := \varphi(1, (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})) \): unipotent element of \( G_\varphi \)
- \( A_{G_\varphi}(u_\varphi) := \pi_0(Z_{G_\varphi}(u)) \).

We have
\[
S_\varphi \simeq A_{G_\varphi}(u_\varphi).
\]

Main idea:
(6) will allow us to use the generalized Springer correspondence for the complex group \( G_\varphi \) in order to understand the structure of the \( L \)-packets for the \( p \)-adic group \( G \).
The role of the generalized Springer correspondence

Generalized Springer variety [Lusztig, Invent. math. 1984]

Let $G$ be a connected reductive group over $\mathbb{C}$, and let
- $P = LU$ parabolic subgroup of $G$
- $u \in G$ and $v \in L$ unipotent elements.

The group $Z_G(u) \times Z_L(v)U$ acts on the variety

$$Y_{u,v} := \{ y \in G : y^{-1}uy \in vU \}$$

by $(g, p) \cdot y := gyp^{-1}$, with $g \in Z_G(u)$, $p \in Z_L(v)U$ and $y \in Y_{u,v}$.

The group $A_G(u) \times A_L(v)$ acts on the set of irreducible components of $Y_{u,v}$ of maximal dimension (i.e. $\dim U + \frac{1}{2}(\dim Z_G(u) + \dim Z_L(v))$). Let $\sigma_{u,v}$ denote the corresponding permutation representation.
**Definition** [Lusztig, Invent. math. 1984]

Let $\rho \in \text{Irr}(A_G(u))$. Then $\rho$ is called **cuspidal** if

$$\langle \rho, \sigma_u, v \rangle_{A_G(u)} \neq 0 \text{ for any unipotent } v \in \mathcal{L} \Rightarrow \mathcal{P} = G,$$

where $\langle , \rangle_{A_G(u)}$ is the usual scalar product on the space of class functions on $A_G(u)$ with values in $\overline{\mathbb{Q}}_\ell$.

**Note:** If $(u, \rho)$ is cuspidal, then $C$ is a **distinguished** (i.e. $C$ does not meet the unipotent variety of $\mathcal{L}$ for any $\mathcal{L} \neq G$). However, in general not every distinguished unipotent class supports a cuspidal representation.

**Theorem** [Lusztig, loc. cit.]

Let $C$ be a unipotent class in $G$ and $\mathcal{E}$ an irreducible $G$-equivariant local system on $C$. The IC-sheaf $F_\rho := IC(C, \mathcal{E}_\rho)$ occurs as a summand of $i_G^{\mathcal{L} \subset \mathcal{P}}(IC(C_{\text{cusp}}, \mathcal{E}_{\text{cusp}}))$, for some triple $(\mathcal{P}, \mathcal{L}, (C_{\text{cusp}}, \mathcal{E}_{\text{cusp}}))$, where $\mathcal{P}$ is a parabolic subgroup of $G$ with Levi subgroup $\mathcal{L}$ and $(C_{\text{cusp}}, \mathcal{E}_{\text{cusp}})$ is a cuspidal unipotent pair in $\mathcal{L}$. Moreover, the triple $(\mathcal{P}, \mathcal{L}, (C_{\text{cusp}}, \mathcal{E}_{\text{cusp}}))$ is unique up to $G$-conjugation.
Definition

Let $\rho \in \text{Irr}(A_{G^\circ}(u))$. The **cuspidal support** of $(u, \rho)$, denoted by $\text{Sc}_{G^\circ}(u, \rho)$, is defined to be

$$\left(\mathcal{L}, (v, \rho_{\text{cusp}})\right)_G,$$

where $v \in \mathcal{C}_{\text{cusp}}$ and $\rho_{\text{cusp}} \leftrightarrow \mathcal{E}_{\text{cusp}}$. (7)

Disconnected complex reductive groups [A.-Moussaoui-Solleveld, 2018]

Let $G$ be a possibly disconnected reductive group over $\mathbb{C}$, with identity component $G^\circ$. Let $u \in U(G)$ and $\rho \in \text{Irr}(A_G(u))$. We observe that $A_{G^\circ}(u) \subset A_G(u)$.

- The pair $(u, \rho)$ is called **cuspidal** if the restriction of $\rho$ to $A_{G^\circ}(u)$ is a direct sum of irreducible representations $\rho^\circ$ such that one (or equivalently any) of the pairs $(u, \rho^\circ)$ is cuspidal.

- We set $\mathcal{T} := Z_{\mathcal{L}}^\circ$ and $\mathcal{M} := Z_G(\mathcal{T})$. The cuspidal support of $(u, \rho)$ is a (well-defined) triple $(\mathcal{M}, v, \rho_{\text{cusp}})_G$, where $\rho_{\text{cusp}}$ occurs in the restriction of $\rho_{\text{cusp}}$ to $A_{G^\circ}(u)$.
Remark

By the Jacobson–Morozov theorem, any unipotent element $v$ of $L$ can be extended (in a unique way up to $Z_L(v)\circ$-conjugation) to a homomorphism of algebraic groups

$$j_v : \text{SL}_2(\mathbb{C}) \to L \text{ satisfying } j_v \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) = v. \quad (8)$$

Definition [A.-Moussaoui-Solleveld, 2018]

An enhanced $L$-parameter $(\varphi, \rho) \in \Phi_e$ is called cuspidal if the following properties hold:

- $\varphi$ is discrete (i.e., $\varphi(W'_F)$ is not contained in any proper Levi subgroup of $G^\vee$),
- $(u_\varphi, \rho)$ is a cuspidal pair in $G_{\varphi}$.

We denote by $\Phi_{e,\text{cusp}}(G)$ the set of $G^\vee$-conjugacy of cuspidal enhanced $L$-parameters for $G$. 
The generalized Springer correspondence allows us to define a cuspidal support map

$$\text{Sc}: \Phi_e(G) \to \bigcup_{L\text{ Levi de } G} \Phi_{e,\text{cusp}}(L).$$  \hspace{1cm} (9)

**Definition of the map Sc**

Let $\varphi: W_F \times SL_2(\mathbb{C}) \to {}^L G$. We define $\varphi_v: W_F \times SL_2(\mathbb{C}) \to Z_{G^\vee}(T)$ by

$$\varphi_v(w, x) := \varphi(w, 1) \cdot \chi_{\varphi, v}(\|w\|^{1/2}) \cdot j_v(x) \quad \text{for all } w \in W_F, x \in SL_2(\mathbb{C})$$

where

$$\chi_{\varphi, v}: z \mapsto \varphi \left(1, \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \right) \cdot j_v \left( \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} \right) \quad \text{for } z \in \mathbb{C}^\times.$$

The cuspidal support of $(\varphi, \rho)$ is defined to be

$$\text{Sc}(\varphi, \rho) := (Z_{G^\vee}(T), (\varphi_v, \rho_{\text{cusp}})).$$  \hspace{1cm} (10)
The cuspidal $G$-relevant enhanced Langlands parameters correspond by the LLC to the irreducible supercuspidal representations of $G$:

$$\text{LLC: } \text{Irr}_{\text{cusp}}(G) \leftrightarrow \Phi_{e,\text{cusp}}(G).$$

### State of art

The cuspidality conjecture is known to hold for all the Levi subgroups (including the groups themselves) of

- general linear groups and split classical $p$-adic groups [Moussaoui, 2017],
- inner forms of linear groups and of special linear groups, and quasi-split unitary $p$-adic groups [A-Moussaoui-Solleveld, 2018],
- the $p$-adic group $G_2$ [A-Xu, 2022],
- pure inner forms of quasi-split classical $p$-adic groups [A-Moussaoui-Solleveld, 2022].
**Definition**

- $L^\vee$ Langlands dual group of a Levi subgroup $L$
- $\mathfrak{X}_{nr}(L^\vee) := \{\zeta : W_F/I_F \to Z_{L^\vee}^0\}$, which acts on the set of cuspidal enhanced $L$-parameters for $L$.
- $\mathfrak{s}^\vee := [L^\vee \rtimes W_F, (\varphi_{\text{cusp}}, \rho_{\text{cusp}})]_{G^\vee}$ the $G^\vee$-conjugacy class of $(L^\vee \rtimes W_F, \mathfrak{X}_{nr}(L^\vee) \cdot (\varphi_{\text{cusp}}, \rho_{\text{cusp}}))$, where $(\varphi_{\text{cusp}}, \rho_{\text{cusp}}) \in \Phi_{e,\text{cusp}}(L)$
- $\mathcal{B}^\vee(G)$ the set of such $\mathfrak{s}^\vee$.
- $\Phi_{e}^{\mathfrak{s}^\vee}(G)$: fiber of $\mathfrak{s}^\vee$ under the map $S_c$.

**Theorem [A.-Moussaoui-Solleveld, 2018]**

The set $\Phi_e(G)$ of $G^\vee$-conjugacy classes of enhanced $L$-parameters is partitioned into *series à la Bernstein* as

$$\Phi_e(G) = \prod_{\mathfrak{s}^\vee \in \mathcal{B}(G^\vee)} \Phi_{e}^{\mathfrak{s}^\vee}(G). \quad (12)$$
A variant of the group $G_\varphi$:
Let $I_F \subset W_F$ be the inertia group of $F$. We define

$$J_{\varphi} := Z_{G^\vee}(\varphi(I_F)).$$

A root system:
Let $s^\vee := [L^\vee \times W_F, (\varphi_{\text{cusp}}, \rho_{\text{cusp}})]_{G^\vee} \in \mathfrak{B}^\vee(G)$. Recall $\mathcal{T} = Z_{\mathcal{L}}^\circ$.
Define $R(J^\circ, \mathcal{T})$ to be the set of $\alpha \in X^*(\mathcal{T}) \setminus \{0\}$ which appear in the adjoint action of $\mathcal{T}$ on the Lie algebra of $J_{\varphi}^\circ$. It can be shown that $R(J^\circ, \mathcal{T})$ is a root system. We denote by $W_{s^\vee}^\circ$ its Weyl group.

An extended finite Weyl group:
Let $W_{s^\vee} := N_{G^\vee}(s^\vee)/L^\vee$. We have $W_{s^\vee} = W_{s^\vee}^\circ \rtimes \Gamma_{s^\vee}$, where

$$\Gamma_{s^\vee} := \{ w \in W_{s^\vee} : w(R(J^\circ, \mathcal{T})^+) \subset R(J^\circ, \mathcal{T})^+ \}.$$
A root datum:

We define a root datum

\[ R_{s^\vee} := (R_{s^\vee}, X^*(T_{s^\vee}), R_{s^\vee}^\vee, X^*_s(T_{s^\vee}^\vee)), \]

where \( T_{s^\vee} \cong s_{L^\vee}^\vee = [L^\vee, (\varphi_{cusp}, \rho_{cusp})]_{L^\vee} \) and

\[ R_{s^\vee} = \{ m_\alpha \alpha : \alpha \in R(J^\circ, T)_{\text{red}} \subset X^*(T_{s^\vee}) \}, \]

with \( m_\alpha \in \mathbb{Z}_{>0} \). The group \( W_{s^\vee} \) acts on \( R_{s^\vee} \).

Weight functions:

We define \( W_{s^\vee} \)-invariant functions

\[ \lambda : R_{s^\vee} \to \mathbb{Q}_{>0} \quad \text{and} \quad \lambda^* : \{ m_\alpha \alpha \in R_{s^\vee} : m_\alpha \alpha \in 2X^*_s(T_{s^\vee}) \} \to \mathbb{Q}. \]
A twisted affine Hecke algebra:

The algebra $\mathcal{H} := \mathcal{H}(G^\vee, s^\vee)$ is defined as

$$\mathcal{H}(G^\vee, s^\vee) := \mathcal{H}(R_s^\vee, \lambda, \lambda^*, q^{1/2}) \rtimes \mathbb{C}[\Gamma_s^\vee, \natural_s^\vee],$$

where $\natural_s^\vee$ is a certain 2-cocycle.

Theorem [A-Moussaoui-Solleveld, 2018]

There exists a canonical bijection

$$\Phi_e^s (G) \rightarrow \text{Irr}(\mathcal{H}(G^\vee, s^\vee))$$

$(\varphi, \rho) \mapsto M(\varphi, \rho)$

with the following properties

1. $\varphi$ is bounded if and only if $M(\varphi, \rho)$ is tempered,
2. $\varphi$ is discrete if and only if $M(\varphi, \rho)$ is an essentially discrete series and the rank of $R_s^\vee$ equals the dimension of $T_s^\vee/\mathcal{X}_{nr}(^LG)$. 
**Theorem**

If $G$ is

1. an inner twist of $\text{GL}_n(F)$ [A-Baum-Plymen-Solleveld, 2019]
2. a pure inner twist of quasi-split classical $p$-adic group [A-Moussaoui-Solleveld, 2022]
3. the group $G_2$ [A-Xu, 2022]

then, for every $s = [L, \sigma]_G \in \mathcal{B}(G)$ such that $L \neq G$

$$\mathcal{H}^s(G) \overset{\text{Morita}}{\sim} \text{Mod}(\mathcal{H}(G, s)) \quad \text{with} \quad \mathcal{H}(G, s) \cong \mathcal{H}(G^\vee, s^\vee)$$

where $s^\vee := [L^\vee \rtimes W_F, \text{LLC}(\sigma)]_{G^\vee}$.

In cases (1) (resp. (2)), the bijection

$$\mathcal{L}^G: \text{Irr}^s(G) \overset{1-1}{\rightarrow} \text{Irr}(\mathcal{H}(G, s)) \overset{1-1}{\rightarrow} \text{Irr}(\mathcal{H}(G^\vee, s^\vee)) \overset{1-1}{\rightarrow} \Phi_e^{s^\vee}(G)$$

coincides with LLC defined by Harris-Taylor (resp. Arthur) for all $s \in \mathcal{B}(G)$ (including the case $L = G$).
Remark

In all the cases (1), (2) et (3), the following diagram is commutative

\[
\begin{array}{ccc}
\text{Irr}^s(G) & \xrightarrow{L^G} & \Phi_e^s(G) \\
\downarrow \text{Sc} & & \downarrow \text{Sc} \\
\text{Irr}^s_L(L) & \xrightarrow{1-1 \text{ LLC}} & \Phi_e^s_L(L)
\end{array}
\]

Conjecture [A.-Moussaoui-Solleveld, 2018]

Such a commutative diagram exists for every $p$-adic group $G$ and all $s \in \mathcal{B}(G)$. 
Thank you very much for your attention!