Modularity for W-algebras, affine Springer Fibres and associated variety.

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Motivation from physics

$$\{\text{4d } N=2 \text{ SCFT}\} \xrightarrow[V]{\text{Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees}} \{\text{Vertex Alg}\}$$

$$T \longmapsto V(T)$$

$$\cup$$

Class S of type ADE

$$\| $$

$$(\text{6d } (2,0) \text{ SCFT on } \mathbb{R}^{3,1} \times C)$$

Coulomb — Gaiotto-Moore-Neitzke: $\mathcal{M}_C$ = Hitchin moduli space

Higgs (Arakawa): Associated variety $X_{V(T)}$

$$\text{generalised Argyres-Douglas theory} \xrightarrow{V} \{W_k(\hat{g}, f)\}$$

$$\mathcal{M}_C = \left\{ (\mathcal{E}, \phi) \;\middle|\; \begin{array}{l} \mathcal{E} \ \ G\text{-bundle on } \mathbb{P}^1 \\ \phi : \mathcal{E} \to \mathcal{E} \otimes \omega_{\mathbb{P}^1} \\ \phi|_0 \sim \frac{f^\vee}{z} + \cdots \\ \phi|_\infty \sim \frac{A}{z^{k^\vee}} + \cdots \end{array} \right\} \Big/ \sim$$

A reg ss.

$W_{-N+\frac{N}{N+M}}(A_{N-1}, \text{principal})$.

Fredrickson–Neitzke 2017: for $W_N(N,M)$ minimal model

$\exists$ bijection $\left\{ \begin{array}{c} \mathbb{C}^\times\text{-fixed point} \\ \text{in Hitchin fibre} \end{array} \right\} \longleftrightarrow \mathrm{Irrep}(W\text{-alg})$

- We estabish such a bijection for arbitrary W-alg at boundary admissible level. (modulo conj). using ASF

Plan:
- explain bijection for affine KM-alg
- modularity of characters in terms of DAHA
- explain the W-alg case.

jt with Dan Xie, Wenbin Yan

- further result / conj about nonadmissible case and associated var.

jt with Wenbin Yan - Qixian Zhao

1. Admissible rep^n

$\mathfrak{g}$ f.d. simple Lie alg

$\hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$ affine Lie alg [talk: $\mathfrak{g}$ simply-laced ; paper: any affine KM]

$V_\kappa = U(\hat{\mathfrak{g}}) \underset{U(\mathfrak{g}[t] \oplus \mathbb{C}K \oplus \mathbb{C}d)}{\otimes} \mathbb{C}_\kappa$ $\forall \kappa \in \mathbb{C}$ s.t. K acts by κ

Universal affine VOA

$L_\kappa =$ simple quotient of V_κ

Fix κ boundary admissible

i.e. $\kappa + h^\vee = \frac{h^\vee}{u}$, for $(u, h^\vee) = 1$, $u \in \mathbb{Z}_{>0}$

$\mathrm{Rep}(L_\kappa)$ inside $\mathcal{O}(\hat{\mathfrak{g}})_\kappa$ is semi-simple, with

simples given by $L(\lambda)$, $\lambda \in \mathrm{Adm}_\kappa = \{$admissible wts of level $\kappa\}$,

where $\lambda \in \mathrm{Adm}_\kappa \overset{\mathrm{df}}{\Longleftrightarrow} \begin{cases} \lambda \text{ reg. dom, i.e. } \langle \lambda + \hat{\rho}, \alpha^\vee \rangle \notin \{0,1,2,\cdots\} \quad \forall \alpha^\vee \in \Phi_{re}^{+,\vee} \\ \Phi(\lambda)^\vee \simeq \Phi^\vee \end{cases}$

$\overset{[KW]}{=} \{ w \cdot \kappa\Lambda_0 \mid w \in W_{ex},\ w(\Pi_u^\vee) \subset \Phi^{+,\vee} \}$

where $\Pi_u^\vee = \{ \alpha_1^\vee, \cdots, \alpha_\ell^\vee, -\theta^\vee + uK \}$

$W_{ex} = W \ltimes P^\vee$, $P^\vee$ coweight lattice

2. Affine Springer Fibre (ASF)

$\mathfrak{g}^\vee$ Langlands dual Lie alg

$G^\vee$ conn. algebraic grp of adjoint type w/ $\mathrm{Lie}(G^\vee) = \mathfrak{g}^\vee$

$\cup$

$B^\vee \supset T^\vee$

$\mathcal{F}\ell = G^\vee((z))/I^\vee$, $I^\vee$ = Iwahori subgroup

$\Omega = \pi_0(\mathcal{F}\ell) \curvearrowright \mathcal{F}\ell$, let $\mathcal{F}\ell^\circ$ be neutral component

$\gamma = \sum_{i=1}^{\ell} e_{\alpha_i^\vee} + z^u f_{\theta^\vee} \in \mathfrak{g}^\vee[z]$ homo. elliptic element.

slope $\frac{u}{h^\vee}$

ASF: $\mathcal{F}\ell_\gamma := \{ gI^\vee \mid \mathrm{Ad}_{g^{-1}}(\gamma) \in \mathrm{Lie}(I^\vee) \}$ f.d. proj var.

$\mathbb{C}^\times \curvearrowright \mathcal{F}\ell_\gamma$ via $\mathbb{C}^\times \longrightarrow T^\vee \times \mathbb{C}^\times_{rot}$

$t \longmapsto (t^{u\rho}, t^{h^\vee})$

$$\mathcal{F}\ell_\gamma^{\mathbb{C}^\times} = \underbrace{\mathcal{F}\ell_\gamma \cap \mathcal{F}\ell^{T^\vee}}_{\{x \in W_{ex} \mid x^{-1}(\Pi_u^\vee) \subset \Phi_+^\vee\}} \subset \underbrace{\mathcal{F}\ell^{T^\vee}}_{W_{ex}}$$

Thm 1 (SXY) The natural map $\mathcal{F}\ell_\gamma^{\mathbb{C}^\times} \longrightarrow \mathrm{Adm}_\kappa$, $x \longmapsto x^{-1} \cdot \kappa\Lambda_0$

yields a bijection $(\mathcal{F}\ell_\gamma^\circ)^{\mathbb{C}^\times} \xrightarrow{\sim} \mathrm{Adm}_\kappa$

Rk 1: Works for any affine Lie algebra $\hat{\mathfrak{g}}$ (including twisted ones)

with $\mathbb{C}^\times$-fixed point in ASF for $(\hat{\mathfrak{g}})^\vee$

Rk2 : [BBAMY] Hitchin fibre $\underset{\text{homeo}}{\simeq}$ Fl_γ°

so matches with physics expectation.

3. Modularity [talk: $\mathfrak{g}$ simply laced, paper: $\hat{\mathfrak{g}}$ untwisted AKM]

$\forall \lambda \in Adm_k$, the renormalised character

$$Ch_\lambda(v) := e^{2\pi i \tau s_\lambda} \, Tr_{L(\lambda)}(e^v)$$

for $v \in Y = \{ (z,\tau,t) := 2\pi i (z + tK - \tau d) \mid z \in \mathfrak{h},\ \tau, t \in \mathbb{C},\ Im(\tau) > 0 \}$

is meromorphic function on Y

Kac-Wakimoto:

$V_{ch} =: span\{ Ch_\lambda \mid \lambda \in Adm_k \} \subset Fun(Y)$

invariant under $SL_2(\mathbb{Z})$-action

and give explicit formulae for matrices for $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

DAHA

$\mathbb{H}_{q,t} = \mathbb{C}_{q,t}$-alg deforming $\mathbb{C}_{q,t}[X_b \mid b \in P^\vee] \otimes \mathbb{C}W \otimes \mathbb{C}[Y_b, b \in P]$

$\mathbb{H}_{q,t} \curvearrowright Pol = \mathbb{C}[X_b, b \in P^\vee]$

$\{E_b\}_{b \in P^\vee}$ renormalised Non-Symmetric MacDonald pol^n

form eigenbasis for $\mathbb{C}[Y]$-acting on Pol.

s.t. $L_f(Y)(\mathcal{E}_b) = f(q^{-b_\#})\,\mathcal{E}_b$, $b_\# = b - u_b^{-1}(\kappa\rho)$.

Assume $t^{h^\vee} = q^{-u}$, q generic

Pol has a finite dim quotient $\mathbb{V}_\gamma$

$\mathrm{Pol} \longrightarrow \mathbb{V}_\gamma$, $\mathcal{E}_b \mapsto 0$ if $b \notin \Sigma_u \subset P^\vee$

X-acts semi-simply on $\mathbb{V}_\gamma = \mathrm{Fun}(\Sigma_u/\Omega)$

V_γ has basis $\{\chi_b\}_{b\in\Sigma_u/\Omega}$

Geometric picture $\mathbb{H}_{q,t} = K(\hat{St}/_{G((t))\rtimes \mathbb{C}^\times_{rot}\times\mathbb{C}^\times}) \curvearrowright K^{\mathbb{C}^\times}(\mathcal{F}\ell_\gamma)_{loc} = \mathbb{V}_\gamma$

Cherednik $PSL_2^c(\mathbb{Z}) \curvearrowright \mathbb{P}\mathrm{Aut}(\mathbb{H}_{q,t})$

$\quad \| $

$\langle \sigma, \tau_+ \mid (\sigma\tau_+)^3 = \sigma^2 \rangle \circlearrowright \mathbb{P}\mathrm{Aut}(V_\gamma)$

with $S_{b,b'} = \mathcal{E}_b(q^{b'_\#})\,\mu(q^{b'_\#})$

T = mult. by Gaussian

Thm 2 (SXY) Set $q = e^{-2\pi i\, h^\vee/u}$ $(\Rightarrow t=1)$

The isom $\mathbb{V}_{ch} \simeq \mathbb{V}_\gamma$ commute with $SL_2(\mathbb{Z})$-act^n.

$(-1)^{\ell(u_b)}\mathrm{Ch}_{\pi_b} \longleftarrow \chi_b$

4. W-algebras

$\mathfrak{g}$ simply-laced.

f has good even grading (otherwise, need Raymond twist)

Assume f regular in a Lévi L (let $\Phi_L^\vee$ = coroots for L)

$W_k(\mathfrak{g}, f) =:$ + - quantum DS-reduction of $\mathbb{L}_k(\mathfrak{g})$ [KRW]

$$\Psi_f^- : \quad \mathbb{L}_k(\mathfrak{g})\text{-mod} \longrightarrow W_k(\mathfrak{g}, f)\text{-mod}$$

$\parallel$ - reduction functor

Conj (KRW, Arakawa) For $\lambda = x \cdot k\Lambda_0 \in Adm_k$

- $\Psi_f^-(L(\lambda))$ is simple if it is nonzero
- $\Psi_f^-(L(\lambda)) \neq 0 \iff x(\Pi_u^\vee) \subset \Phi_+^\vee \setminus \Phi_L^\vee$
- $\Psi_f^-(L(\lambda)) = \Psi_f^-(L(\lambda')) \iff \lambda \in W_L \circ \lambda'$, where W_L = Weyl gp for L

All simple modules are obtained in this way.

Conj proved for
- f principal
- any f in type A
- exceptional (f, u).

Spaltenstein variety: let $L^\vee \subset G^\vee$ Lévi subgp defined by $\Phi_L^\vee$

$P^\vee \subset G^\vee$ parabolic subgp gen. by $L^\vee$ and $B^\vee$.

$$\begin{array}{ccccc} G^\vee(K) & > & G^\vee(\mathcal{O}) & \xrightarrow{ev} & G^\vee \\ & & \cup & & \cup \\ & ev^{-1}(P^\vee) = & \mathbb{P}^\vee & \longrightarrow & P^\vee \end{array}$$

$$\mathbb{C}^\times \curvearrowright \mathcal{F}l_\gamma^{\mathbb{P}^\vee} := \{ g\mathbb{P}^\vee \mid g^{-1}\gamma g \in \mathrm{Lie}(\mathrm{rad}(\mathbb{P}^\vee)) \} \subset G^\vee(K)/\mathbb{P}^\vee$$

Thm [SXY] Assuming conjectures hold. Then the bijection

$$\mathrm{Irr}(\mathbb{L}_k) \simeq (\mathcal{F}l_\gamma^\circ)^{\mathbb{C}^\times} \qquad (*)$$

induces a bijection

$$\mathrm{Irr}(W_k(\mathfrak{g}, f)) \simeq (\mathcal{F}l_\gamma^{\mathbb{P}^\vee, \circ})^{\mathbb{C}^\times}$$

Cor: $$|\mathrm{Irr}(W_k(\mathfrak{g}, f))| = \frac{1}{|W_L|} u^{\ell - j} \prod_{i=1}^{j} (u - e_i)$$

where $e_1, \cdots, e_j$ are exponents of W_L, $\ell = \mathrm{rk}\,\mathfrak{g}$

Thm: [SXY] let $\mathbb{V}_k^f = \{ \mathrm{Ch}_L \mid L \in \mathrm{Irr}(W_k(\mathfrak{g}, f)) \} \circlearrowleft SL_2(\mathbb{Z})$

$$\mathbb{V}_\gamma^{P^\vee} = e_L^-(\mathbb{V}_\gamma) \curvearrowleft e_L^- H_{q,t} e_L^-$$

sgn idempotent

$$\circlearrowleft \quad \circlearrowright$$

$$PSL_2^c(\mathbb{Z}).$$

Then $\mathbb{V}_k^f \simeq \mathbb{V}_\gamma^{P^\vee}$ defined by the bijection $(*)$

is $SL_2(\mathbb{Z})$-equivariant.

Rk: Such identifications was estabished previously for

- Virasoro minimal model by Koczac-Shakirov-Yan.
- $W_N(N,M)$ minimal model by Gukov-Koroteev-Nawata-Pei-Saberi.

5. Associated variety [jt with Yan-Zhao]

Physics th. expect $\mathcal{M}_{Higgs}$ has finitely many symplectic leaves.

Q: For which κ. L_κ is quasi-lisse ? ($\Leftrightarrow X_{L_\kappa}$ satisfy $\uparrow$)

L_κ quasi-lisse $\Rightarrow$ it has finitely many highest wt modules.

not always true.

Expect: $\exists$ bijection $\pi_0(\mathcal{Fl}_\gamma^{\mathbb{C}^\times}) \longleftrightarrow \mathrm{Irr}(L_\kappa)$

for γ homo. elliptic. $\Rightarrow$ $\kappa + h^\vee = \frac{m}{u}$,

m elliptic reg. number

$(u,m)=1$. (non-cedm).

let $\Phi: \underline{W} \to \underline{N}^\vee$ Lusztig map (RT$_{min}$ by Yun)

$\gamma \rightsquigarrow [w_\gamma] \longmapsto \Phi[w_\gamma] := \mathcal{O}_\gamma$

$d: \mathcal{N}^\vee \longrightarrow \mathcal{N}$ Barbasch-Vogan-Lusztig-Spaltenstein duality

Conj (SYZ) let $\kappa + h^\vee = \frac{m}{u}$, m elliptic regular, $(u,m)=1$.

then $X_{L_\kappa} = d(\mathcal{O}_\gamma)$

Examples of known cases :

0) Admissible case $(u, h^\vee) = 1$

e.g. (type A)

- $u > h^\vee$, $\mathbb{O}_x = \{0\}$, $d(\mathbb{O}_x) = \mathcal{N}$

- $u < h^\vee$, let $\lambda^{h^\vee, u}$ = partition with u parts, s.t. each part is either $\lfloor \frac{h^\vee}{u} \rfloor$ or $\lceil \frac{u}{h^\vee} \rceil$

$$d(\mathbb{O}_{\lambda^{h^\vee, u}}) = \mathbb{O}_{(\lambda^{h^\vee, u})^t} = [u, \cdots, u, s] \qquad \text{(Jakob-Yun)}$$

In this case X_{L_k} computed by Arakawa

Works for all cases

1) Sub-Coxeter case $(u = 1)$

Deligne exceptional sequence.

$$D_4 \subset E_6 \subset E_7 \subset E_8.$$

m	4	9	14	24
k	-2	-3	-4	-6
$h^\vee$	6	12	18	30

level $k + h^\vee = m$

$\mathbb{O}_x$ = subregular orbit $\quad d(\mathbb{O}_x)$ = minimal orbit
$= X(L_k)$ by [Arakawa-Moreau].

2) $L_{2-n}(D_n)$, with n even, $m = n$ $\qquad$ Coxeter $2n-2$

$k = m - (2n-2) = 2-n$

$$X_{L_{2-n}(D_n)} = \overline{\mathbb{O}_{(2^{n-2}, 1^4)}} \qquad [AM]$$

$$\mathbb{O}_x = [n+1, n-1] , \qquad d\mathbb{O}_x = \mathbb{O}_{(2^{n-2}, 1^4)}$$

Rk: If n odd, $m = n$ is not elliptic regular, $L_{2-n}(D_n)$ is not quasi-lisse