Pre-cuspidal families and indexing of Weyl group representations.

1)

W - Weyl group, S - simple reflections

$\mathcal{R}_W$ - $\mathbb{C}$-vector space with basis $\mathrm{Irr}(W)$. For $E \in \mathrm{Irr}(W)$

let $E(q)$ = corresp irr rep. of $G(\mathbb{F}_q)$ with Weyl gp W,

$\dim E(q) = \frac{1}{m} q^{a_E} + \text{higher} \in \mathbb{Q}[q]$.

For $I \subset S$, $W_I \subset W$ subgp. gener by I, Define $J_{W_I}^{W} : \mathcal{R}_{W_I} \to \mathcal{R}_W$ (linear):

$$\underset{\underset{\mathrm{Irr}(W_I)}{\uparrow}}{E_1} \longrightarrow \sum_{\substack{E \in \mathrm{Irr}\, W \\ a_E = a_{E_1}}} (E_1 : E|_{W_I})\, E.$$

Define a subset $\mathcal{C}on_W \subset \mathcal{R}_W$ by ind. on $|W|$. If $W=\{1\}$, $\mathcal{C}on_W = \{1\}$. Assume $W \neq \{1\}$

$\mathcal{C}on_W = \{ J_{W_I}^W(\rho) ; I \subsetneq S, \rho \in \mathcal{C}on_{W_I} \} \cup$ (same) $\otimes$ sign.

If $E_1 \in \mathrm{Irr}\, W$, $E_2 \in \mathrm{Irr}(W)$ we say $E_1 \sim E_2$ if $\exists\, E_3 \in \mathrm{Irr}(W)$ s.t. E_1, E_3 appear in the same $\sigma \in \mathcal{C}on_W$ and E_2, E_3 appear in the same $\sigma' \in \mathcal{C}on_W$.

This is an equivalence relation. Eq. classes = "families". Let $\Phi(W)$ = set of families

In type A, families are single irr. reps

If $c \in \Phi(W)$ then $c \otimes \text{sign} \in \Phi(W)$.

If $I \subset S$, $c_1 \in \Phi(W_I)$ there is a unique $c \in \Phi(W)$ such that

$\forall\, E_1 \in c_1$, $J_{W_I}^W(E_1)$ is a lin. comb. of reps. in c. We set $c = J_{W_I}^W(c_1)$.

3) For a finite group Γ let $M(\Gamma) = \{(x, \rho) ; x \in \Gamma, \rho \in \mathrm{Irr}\, Z_\Gamma(x)\} / \Gamma\text{-conj}$

$=$ irred. Γ-equivar. vector bundles on Γ. (conj. action)

For any $c \in \Phi(W)$ there is a finite group Γ_c and a natural embedding $c \subset M(\Gamma_c)$. If W is irreducible Γ_c is in the following list

$$A = \begin{cases} \mathbb{Z}_2^n \text{ with basis } e_1, e_2, \ldots e_n \\ \mathbb{Z}_2^{n+1} \text{ with basis } e_1, e_2, \ldots e_{n+1} \text{ modulo } \langle e_1 + e_2 + \ldots + e_{n+1} \rangle \\ S_n \quad n = 2, 3, 4, 5. \\ S_3', S_2'. \end{cases}$$

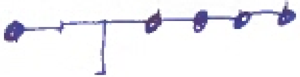
4) We say that $c \in \Phi(W)$ is <u>smoothly induced</u> from $c_1 \in \Phi(W_I)$, $I \neq S$ if $E_1 \in c_1 \Rightarrow J_{W_I}^{W}(E_1) \in \mathrm{Irr}\, W$ and $E_1 \to J_{W_I}^{W}(E_1)$ is a bijection $c_1 \xrightarrow{\sim} c$.

We say that $c \in \Phi(W)$ is <u>cuspidal</u> if neither c nor $c \otimes \mathrm{sign}$ is smoothly induced. Assume W irred, $c \in \Phi(W)$ cusp. (<u>not anomalous</u> i.e. $W \neq E_7$). Set $rk(c) = |S| \bmod 2$

$$\Sigma_c = \left\{ (I, c_1) \,\middle|\, \begin{array}{l} I \subset S \\ |I| = |S|-1 \end{array}, c_1 \in \Phi(W_I),\ c = J_{W_I}^{W}(c_1),\ \begin{array}{l} \text{any cusp. component of } c_1 \\ \text{has } rk = rk(c) \bmod 2 \end{array} \right\}$$

c_1 is said to be a <u>pre-cuspidal family</u> attached to c

2) $c \in \Phi(W)$, W irred ($\neq E_7$)	I^* in $(I^*, c_1) \in \Sigma_c$	Γ_c
B_{k^2+k} $(k \geq 2)$	$B_{k^2+k-1}, B_{k^2+k-2}A_1, \dots, B_{k^2-k}A_{2k-1}$	$(\mathbb{Z}/2)^k$
D_4	$A_3, A_3, A_3, A_1A_1A_1$	$\mathbb{Z}/2$
D_{k^2} $(k \geq 3)$	$D_{k^2-1}, D_{k^2-2}A_1, \dots, D_{k^2-2k+2}A_{2k-3}$	$(\mathbb{Z}/2)^{k+1}/(\mathbb{Z}/2)$
E_6	$D_5, D_5, A_2A_2A_1$	S_3
E_8	$E_7, E_6A_1, D_7, D_5A_2, A_4A_3$	S_5
F_4	B_3, B_3, A_2A_1, A_2A_1	S_4
G_2	A_1, A_1	S_3'



5) Let G be a red. gp$/\mathbb{C}$ with conn. centre with Weyl group W (irred.)

Let $un(G)$ = set of unipotent classes of G. There is a canonical imbedding

$j_G : \Phi(W) \subset un(G)$ such that for $c \in \Phi(W)$, $\frac{Z_G(u)}{Z_G(u)^{\circ}} = \Gamma_c$ for $u \in j_G(c)$.

Assume $c \in \Phi(W)$ is cuspidal and $(I, c_1) \in \Sigma_c$. It is known that $j_G(c)$ is <u>induced</u> from $j_L(c_1)$ in the sense of [L-S 79] where P is a parabolic of G (of type I) with Levi quotient $P \xrightarrow{\pi} L$ so that W_I is the Weyl group of L.

7) ~~[illegible]~~ By definition, $V = \pi^{-1} j_L(c_1) \cap j_G(c)$ is open dense in $\pi^{-1} j_L(c_1)$. Let $v \in V$. We can form

$$\Gamma_{c_1} = \frac{Z_L(\pi(v))}{Z_L(\pi(v))^\circ} \xleftarrow[\text{surj [L-S]}]{A} \frac{Z_P(v)}{Z_P(v)^\circ} \xrightarrow[\text{inj [L-S]}]{} \frac{Z_G(v)}{Z_G(v)^\circ} = \Gamma_c$$

Let $\Gamma'' = Z_P(v)/Z_P(v)^\circ$, $\Gamma' = \ker A$

We have $(\Gamma', \Gamma'') \in Z_{\Gamma_c}$ where for a finite group Γ, Z_Γ is the set of pairs $(\Gamma' \subset \Gamma'')$ of subgroups of Γ with Γ' normal in Γ''.

$\partial)$

The pairs $(\Gamma' < \Gamma'')$ obtained in this way from various $(I, c_1) \in \Sigma_c$ for $c \in \Phi(W)$ cuspidal form a subset x_{Γ_c} of Z_{Γ_c}

Recall: $\mathbb{A}$ = collection of finite groups associated to various $c \in \Phi(W)$ or equivalently to various cuspidal $c \in \Phi(W)$ $(W$ irred$)$

We define ~~by~~ a subset $X_\Gamma \subset Z_\Gamma$ for $\Gamma \in \mathbb{A}$ by induction on $|\Gamma|$.

If $\Gamma = \{1\}$ then $X_\Gamma = (\{1\}, \{1\})$. Assume that $\Gamma \neq 1$. Let $(\Gamma', \Gamma'') \in x_\Gamma$ we have $(\Gamma', \Gamma'') \neq (1, \Gamma)$ hence $|\Gamma''/\Gamma'| < |\Gamma|$ so that $X_{\Gamma''/\Gamma'}$ is known by ind.

9) Let $(\Gamma'_1, \Gamma''_1) \in X_{\Gamma''/\Gamma'}$. Take inverse images under $\Gamma \to \Gamma/\Gamma'$.
The resulting pairs (for various $(\Gamma', \Gamma'') \in \mathcal{X}_\Gamma$, $(\Gamma'_1, \Gamma''_1) \in X_{\Gamma''/\Gamma'}$) form a set X^0_Γ. Assuming that Γ is not S_3, S_4, S_5 we define

$$X_\Gamma = X^0_\Gamma \cup (1, \Gamma)$$

In the case where Γ is S_3, S_4, S_5 we need to add more elements to X^0_Γ. We will consider only the case of S_5.

10) Example (S_5) $\quad S_3 \leftarrow S_3 S_2 \hookrightarrow S_5$, $S_2 \leftarrow S_3 S_2 \rightarrow S_5$, $S_2 \leftarrow Dih_8 \longrightarrow S_5$, $1 \leftarrow S_4 \hookrightarrow S_5$, $1 \leftarrow S_5 \hookrightarrow$

x_{S_5} : $\quad (S_2 \subset S_3 S_2) \quad (S_3 \subset S_3 S_2) \quad (S_2 S_2 \subset Dih_8) \quad (S_4 \subset S_4) \quad (S_5 \subset S_5$

$X^{\circ}_{S_5}$: $(S_5 \subset S_5)$, $(S_3 S_2 \subset S_3 S_2)$, $(S_4 \subset S_4)$, $(Dih_8 \subset Dih_8)$, $(S_2 S_2 \subset S_2 S_2)$, $(S_3 \subset S_3)$, $(S_2 \subset S_2)$

$(S_3 \subset S_3 S_2)$, $(S_2 \subset S_3 S_2)$ $(S_2 S_2 \subset Dih_8)$, $(S_2 \subset S_2 S_2)$

To this on has to add all $(1, H)$ with $(H, H) \in X^{\circ}_{S_5}$, $H \in A$ (i.e. $H \neq Dih_8$). There are

$(1 \subset S_5)$, $(1 \subset S_3 S_2)$, $(1 \subset S_4)$ $(1 \subset S_2 S_2)$ $(1 \subset S_3)$ $(1 \subset S_2)$ $\quad$ (Total: 17 pairs)

one obtains X_{S_5}.

11)

Γ finite group

$\mathbb{C}[M(\Gamma)] = \mathbb{C}$ - vector sp. with basis $M(\Gamma) = \mathbb{C} \otimes K_\Gamma(\Gamma)$ equiv. vector bundle

For $H \subset \Gamma$ subgr, define linear map $i_{H\Gamma} : \mathbb{C}[M(H)] \to \mathbb{C}[M(\Gamma)]$

$$(x, \sigma) \to \sum_{\tau \in \operatorname{Irr} Z_\Gamma(x)} (\sigma : \tau|_{Z_H(x)}) (x, \tau)$$

For $H \subset \Gamma$ normal subgr., define linear map $\pi_{H,\Gamma} : \mathbb{C}[M(\Gamma/H)] \to \mathbb{C}[M(\Gamma)]$

as inverse image in equiv. K-theory under $\Gamma \to \Gamma/H$

For $H \subset H' \subset \Gamma$ subgroups, H normal in H', define linear map $s_{H,H'} : \mathbb{C}[M(H'/H)] \to \mathbb{C}[M(\Gamma)]$

as

$$\begin{tikzcd} \mathbb{C}[M(H'/H)] \arrow[dr, "\pi_{H,H'}"'] \arrow[rr] & & \mathbb{C}[M(\Gamma)] \\ & \mathbb{C}[M(H')] \arrow[ur, "i_{H',\Gamma}"'] & \end{tikzcd}$$

We have $R_W = \bigoplus_{c \in \Phi(W)} R_c$, R_c has $\mathbb{C}$-basis c. We have $R_c \subset \mathbb{C}[M(\Gamma_c)]$ since $c \subset M(\Gamma_c)$.

Assume W irr. $\neq E_7$. Define $X_{\Gamma_c} \longrightarrow \mathbb{C}[M(\Gamma_c)]$.
$c \in \Phi(W)$

$(\Gamma' \subset \Gamma'') \longrightarrow s_{\Gamma', \Gamma''}(\underbrace{1, 1}_{\in M(\Gamma''/\Gamma')})$

Thm. The image of this map is contained in R_c and is a basis B_c of R_c. This basis is related to the standard basis of R_c by an upper triang. matrix with entries in $\mathbb{N}$ and 1 on diagonal.

Cor. $c \leftrightarrow B_c \leftrightarrow X_{\Gamma_c}$ canonically. Hence the reps in c are indexed by the pairs $(\Gamma' \subset \Gamma'')$ in X_{Γ_c}.

Cor. The set c has a canonical partial order. There is a unique minimal element = the special repres.