Rigid inner forms / fcn fields

Goal: (loc) For $\underbrace{\boxed{\varphi}}$: $\underbrace{W_F}_{\text{loc. field}} \times SL_2(\mathbb{C}) \to {}^L G$

param. L-packet $\underbrace{\Pi_\varphi}$ in terms of φ

(Gold) For ψ a generic Arthur parameter, describe Π_ψ (in terms of Π_{ϕ_ψ}, ψ); give (conj.) formula for mult. of $\pi \in \Pi_\psi$ in discrete spectrum of G

$$\psi : \underbrace{\overbrace{L_F}^{W_F \times SL_2(\mathbb{C})}}_{\text{hypothetical dual gp}} \to {}^L G$$

Pure inner twists   F nonarch. local

- $\exists$ parametrization of Π_φ for G^* qsplit

- For general G, param. using (G^*, ψ) qsplit inner twist via $[\psi] \in H^1(F, G_{ad}) \to \pi_0(Z(\hat{G}_{sc})^{\Gamma})^*$

Problem: (vogan) $\exists$ aut (G^*, ψ) permuting Π_ψ

E.g. $G = SL_2/\mathbb{R}$, $(G^*, \psi) = \boxed{(SL_2, id)}$, $Ad(g)$, $g = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$\in PGL_2(\mathbb{R})$ this switches holo, anti-holo. disc. series rep $\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \in SL_2(\mathbb{C})$

$\underline{\text{Soln}}$: (ABV) $(G^*, \underline{\psi}) \overset{\text{enriching}}{\longmapsto}$

(G^*, z), $\underline{z} \in Z^1(F, G) \longmapsto$

$\psi \in Z^1(F, G_{ad})$

$\underline{\text{pure inner twist}}$

$\underline{\text{But}}$ since $H^1(F, G) \longrightarrow H^1(F, G_{ad})$ isn't surjective ($H^1(\mathbb{R}, SL_2) = \{*\}$)

Rigid inner twists (F loc, glob. FF)

Want: Refinement of IT's capturing all IT's

- $Z^1(F, G) \longleftrightarrow \boxed{\text{G-torsors on } Sch/F}$

Replace Sch/F, $\boxed{Kal} \to Sch/F$

gerbe corresp. $\boxed{[a] \in H^2_{fppf}(F, u)}$

u profinite gp $/F$

$$\left\{ G\text{-torsors}^{\exists} \text{ on } \boxed{\mathrm{Kal}} \;\middle|\; \exists\, \underline{x^{G}{}_{\mathrm{Gad}}} \text{ descends to } F \right\} / \cong \;\longrightarrow\; \underbrace{H^1(F, \mathrm{Gad})}$$

$\parallel$

"basic" G-torsors on Kal

$\longleftrightarrow$ $H^1(F, G)$

Choose Kal s.t. this map is surjective

What's a gerbe? • Stack $\mathcal{E} \to Sch/F$

fib. in gpoids s.t. any two objects loc. isom

• E.g. 1 $\underline{A}$ comm. gp sch/F,

$\boxed{B_F(\underline{A})} \to Sch/F$; fiber/U

= $\underline{A}_U$-torsors

$\{G\text{-torsors on } B_F(\underline{A})\} =$

$\{G\text{-torsors}/F \text{ w/ equiv. } \underline{A}\text{-action}\}$

E.g. 2: Fix $a \in \underline{A}(\overline{F}^{\otimes_F 3})$

Čech 2-cocycle [wrt cover $\operatorname{Spec}(\overline{F}) \to \operatorname{Spec}(F)$]

$\mathcal{E}_a \to \mathrm{Sch}/F$; fiber$/U = (T, \varphi)$

T is $A_{\overline{F}}$-torsor and $\varphi : p_2^* T \xrightarrow{\sim} p_1^* T$ a-twisted

Check: If $a = 1$, then $\mathcal{E}_a = B_F(\underline{A})$
(fpqc descent)

$$\{G\text{-torsors on } \mathcal{E}_a\} \longleftrightarrow$$

$$\left\{ \underline{A}\text{-equiv. } G_{\bar F}\text{-torsors w/ } \underbrace{a\text{-twisted}} \text{ gluing map } \psi \right\}$$

Fact: $[a] \longmapsto [\mathcal{E}_a]$ gives bij.

$$\tilde H^2(\bar F/F, \underline{A}) \xrightarrow{\sim} \left\{ \underline{A}\text{-gerbes split } / \bar F \right\}$$

Our gerbes: u profinite / F;

$H^2_{fppf}(F, u)$ has canon. class $[a]$,

set $\mathrm{Kal} = \mathcal{E}_a$

$\boxed{H^1_{bas}(\mathrm{Kal}, G) \;\; \xrightarrow{} H^1(F, G_{ad})}$ $\leftarrow \left\{ \begin{array}{c} \text{basic } G\text{-torsors} \\ \text{on Kal} \end{array} \right\} / \simeq$

$= \left\{ \underline{A}\text{-equiv. } G_F\text{-torsors } \middle| \begin{array}{l} \underline{A} \curvearrowright \text{ via} \\ f: \underline{A} \longrightarrow Z(G) \end{array} \right\}$
a-twisted Ψ

Rigid inner twists $\{(G, \Psi)\} \longleftrightarrow \{(G, \mathcal{J})\}$, $\mathcal{J} \mapsto \Psi$

__What is u?__ Have ES

$$\begin{array}{ccccc}
H^1(K_{al}, G) & \longrightarrow & H^1(K_{al}, G_{ad}) & \longrightarrow & \underbrace{H^2(K_{al}, Z(G))}_{=0} \\
\cup & & \cup & & \\
H^1_{bas}(K_{al}, G) & \boxed{\twoheadrightarrow} & H^1(F, G_{ad}) & &
\end{array}$$

__ETS:__ $H^2(K_{al}, M) = 0 \quad \forall$ multiplicative $\boxed{M}$

Loc: $u = \varprojlim_{E/F,\, n} \underbrace{\dfrac{\mathrm{Res}_{E/F}(\mu_n)}{\mu_n}}$

Glob: $u = \varprojlim_{E/F,\, n,\, S} \dfrac{\mathrm{Res}_{E/F}(\mu_n^S)/\mu_n^S}{\underbrace{\mathrm{Res}_{E,S}(\mu_n)/\mu_n}_{\prod_{v \in S} \mathrm{Res}_{E_v/F}(\mu_n)}}$

$H^2_{\mathrm{Ell},F}(F, u_{\mathrm{loc}}) = \hat{\mathbb{Z}}$, $[a] = -1$,

Duality: For nonarch loc. F,

Kottwitz: $\boxed{H^1(F,G) \xrightarrow{\sim} \pi_0(Z(\hat{G})^{\Gamma})^*}$

(Kaletha, D) $H^1_{bas}(\mathrm{Kal}, G) \longleftrightarrow$???

Loc: $\bar{G} := G/Z_{der}$ $\exists$ func canon.

$H^1(F,G) \subseteq \boxed{H^1_{bas}(\mathrm{Kal}_v, G) \xrightarrow{\sim} \pi_0(Z(\hat{\bar{G}})^{+,v})^*}$

$Z(\hat{\bar{G}})^{+,v} = \hat{\bar{G}} \times_{\hat{G}} Z(\hat{G})^{\Gamma_v}$

Glob: $\exists$ "localization"

$$\mathrm{Kal}_v \xrightarrow{\mathrm{loc}_v} \mathrm{Kal}_{glob}$$

Thm: (Kaletha, D)

Map $\boxed{H^1_{bas}(\mathrm{Kal}_{glob}, G)} \to \bigoplus_v H^1_{bas}(\mathrm{Kal}_v, G)$

well-defined, $\underbrace{}_{(\mathrm{loc}_v^*)}$

image $\longleftrightarrow$ kernel of $\left[\bigoplus_v \pi_0(Z(\hat{G})^{+,v})^* \to \boxed{\pi_0(Z(\hat{G})^+)^*}\right]$

Conjectures: Loc: Fix G^*

qsplit, φ_v temp. L-param, $\underline{w}$

Whittaker datum, set

$$\Pi_{\varphi_v} = \left\{ \underline{\pi_v} = \boxed{(G, J_v, \pi_v)} \;\middle|\; (G, J_v) \text{ RIT of } G,\ \pi_v \in \Pi_{\varphi_v}(G) \right\}_{/\cong}$$

"compound L-packet"

<u>Conj:</u> (Kaletha, D) $\exists$ comm. diag

$$\begin{tikzcd}
\Pi_{\varphi_v} \arrow[r, "\boxed{\iota_{\underline{w}}}", "\sim"'] \arrow[d] & \mathrm{Irr}(\pi_0(S^+_{\varphi_v})) \arrow[d, "\text{restriction}"] \\
H^1_{\mathrm{bas}}(\mathrm{Kal}_v, G) \arrow[r, "\text{loc. duality}"'] & \pi_0(Z(\hat{\bar{G}})^{+,v})^*
\end{tikzcd}$$

$\underline{\pi_v} \mapsto \iota_v$

$$S^+_{\varphi_v} = \hat{\bar{G}} \times_{\hat{G}} Z_{\hat{G}}(\varphi)$$

Glob: Fix ψ generic Arthur parameter, ~~gen~~

(G, ψ) IT of quasisplit G^*,

$\underline{\omega}$ whittaker

Def: $\{(G, \mathfrak{J}_v)\}_v$ RIT's enriching (G, ψ_v) **coherent** if

$\exists\ \mathfrak{J} \in H^1_{bas}(\mathcal{E}_{glob}, G^*)$ s.t. $\mathfrak{J}_v = loc_v^{\wedge}(\mathfrak{J})$

- ~~$\Pi_\psi(G)$~~ Fix coherent family $(G, \boxed{\mathcal{J}_v})$ for (G, ψ)

$$\boxed{\Pi_\psi(G)} = \{ \otimes'_v \pi_v \mid \pi_v \in \underline{\Pi_v} = (G, \boxed{\mathcal{J}_v}, \pi_v), \ \underbrace{\langle w_{u,v}(\pi_v) = 1}_{} \text{ a.a. } v \}$$

<u>Prop:</u> $\Pi_\varphi(G)$ consists of irred. admis. reps of $G(\mathbb{A})$

[Kaletha, D, Taibi]

~~Pf:~~

Thm: (Kaletha, D)

$\exists$ pairing (canon) $\boxed{\underline{S}_\varphi \times \Pi_\varphi(G) \to \mathbb{C}}$

glob. analogue of $S_{\varphi_v}^+$, $\hookrightarrow Z_{\hat{G}}(\varphi)$

s.t. $m(\varphi,\pi) := |\underline{S}_\varphi|^{-1} \sum_{\underline{S}_\varphi} \langle s,\pi\rangle \langle -,\pi\rangle \in \mathbb{Z}$

then mult. of $\pi \in \Pi_\varphi$ in disc. spec. of G $= \sum_\varphi m(\pi,\varphi)$

$\mathfrak{X}$ $\quad$ Bun(G) $\quad$ X = FF Fargues

G-torsors on Kott $\longleftrightarrow$

Kal

~~Bun(G)~~

New preprint by Fargues $\widetilde{\mathfrak{X}} \longrightarrow X$

Stacks

V.B on $\mathfrak{X}$ $\longleftrightarrow$ extended isocrystals $\longleftrightarrow$

~~Ess.~~ G-torsors on Kal