Hessenberg varieties and the geometric modular law

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G simple algebraic group $/ \mathbb{C}$

$T \subset B$ maximal torus, Borel subgroup

W Weyl group, $\Phi^+ \subset \Phi$ pos. roots $\subset$ roots

Lie algebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ of $T \subset B \subset G$

$\mathfrak{n}$ nilradical of $\mathfrak{b}$.

$\mathcal{U} = \{$ subspaces of $\mathfrak{n}$, stable under action of $B \}$

"B-stable ideals" "B-stable subspaces"

$$I \in \mathcal{U} \rightsquigarrow I = \bigoplus_{\alpha \in I_\Phi} \mathfrak{g}_\alpha$$

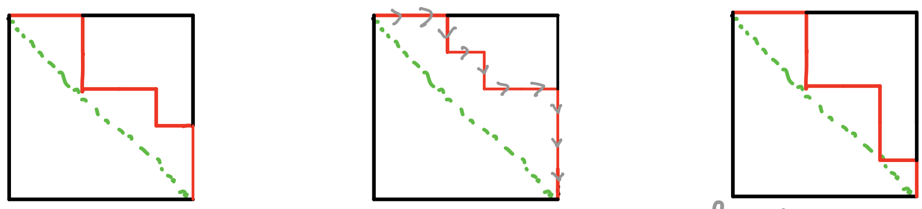
$I_\Phi \subset \Phi^+$ has property that $\alpha \in I_\Phi, \beta \in \Phi^+, \alpha + \beta \in \Phi$
$\Rightarrow \alpha + \beta \in I_\Phi$

Type A_2

$\mathfrak{ad}$ $\quad \begin{bmatrix} & * & * \\ & & * \\ & & \end{bmatrix} \quad \begin{bmatrix} & * & * \\ & & \\ & & \end{bmatrix} \quad \begin{bmatrix} & & \times \\ & & \times \\ & & \end{bmatrix} \quad \begin{bmatrix} & & \times \\ & & \\ & & \end{bmatrix} \quad 0$

Type A_{n-1}: $\mathfrak{ad} \xleftrightarrow{1-1}$ $\boxed{\text{Dyck Path of length } 2n}$

Some $n=5$ cases



eeseseeesss

for this example, always hit diagonal

"Parabolic Case"

$$\#\text{ Dyck paths of length } 2n = \text{Catalan}\,\# = \frac{1}{n+1}\binom{2n}{n}$$

1, 2, 5, 14, 42, 132, . . .

q-version

$n=3$

$1+2q+q^2+q^3$

$1+q^2+q^3+q^4+q^6$

$$\#\,\mathfrak{ad} = \frac{\prod_{i=1}^{n}(h+d_i)}{|W|} \quad \text{(Cellini-Papi)}$$

n = rank

h = Coxeter #

d_i = fundamental degrees

In E_8, $\#\,\mathfrak{ad} = 25{,}080$ "W-Catalan number"

Given $I \in \mathfrak{ad}$, we can form a vector bundle over G/B, flag variety

$$G\times^B I = \left\{ (g, X) \;\middle|\; \begin{matrix} g\in G \\ X\in I \end{matrix} \right\} \Big/ (g,X)\sim(gb, b^{-1}.X)$$

$$\downarrow$$

$$G/B$$

This vector bundle has a map to $\mathfrak{g}$:

$$\pi_I : G\times^B I \longrightarrow \mathcal{N} \qquad \text{Proper map, } G\text{-equivariant}$$

$$(g, X) \longrightarrow g.X$$

$\mathcal{N}$ = nilpotent cone

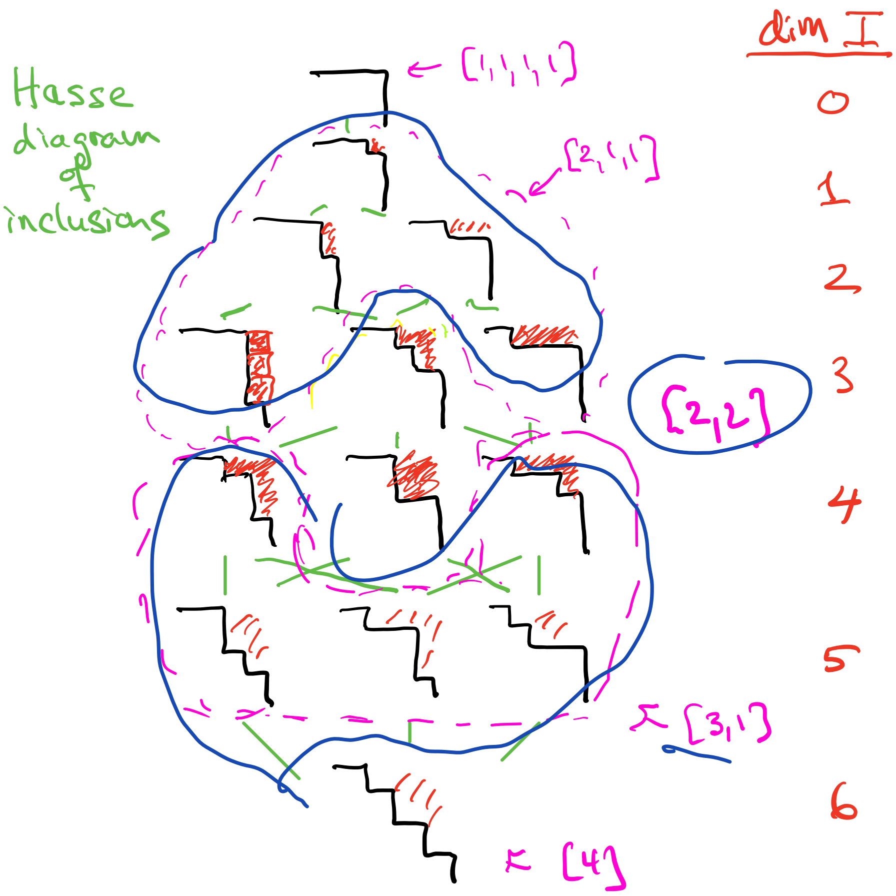
$\operatorname{im}\pi_I = \overline{\mathcal{O}_I}$, closure of a nilpotent orbit $\mathcal{O}_I$

<u>Remark</u>: $I \subseteq J \Rightarrow \mathcal{O}_I \subseteq \overline{\mathcal{O}_J}$

$[I] := \{ J \in \mathcal{I} \mid \mathcal{O}_J = \mathcal{O}_I \}$

Then inclusion descends to partial order on $\{[I]\}$.

<u>Theorem</u>: $\mathcal{O}_I \leq \mathcal{O}_J \Longleftrightarrow [I] \leq [J]$.



$GL_4(\mathbb{C})$

14 subspaces

<u>Def</u>: For $I \in \mathcal{I}$, $e \in \mathcal{N}$, we call $\pi_I^{-1}(e)$ a <u>Hessenberg</u> <u>variety</u>

$$\pi_I^{-1}(e) = \{ gB \in G/B \mid g^{-1}e \in I \} \subseteq \mathcal{B} = G/B$$

Write $\mathcal{B}_e^I$ for this fiber.

$A(e) := Z_G(e)/Z_G^\circ(e)$ component gp of $e \curvearrowright H^*(\mathcal{B}_e^I)$

$\boxed{\text{Case of } I = \eta}$ $\quad G \times^B \eta \longrightarrow \mathcal{N}$ Springer resolution

Fibers = Springer fibers. $\mathcal{B}_e$

Theorem: ① $H^{2i+1}(\mathcal{B}_e, \mathbb{Z}) = 0$ De Concini–Lusztig–Procesi (DLP) '88

② $H^{2i}(\mathcal{B}_e, \mathbb{Z})$ no torsion

③ $H^{2i}(\mathcal{B}_e) \cong i^{\text{th}}$- chow group

Rm 142

Proof Idea: $e \rightsquigarrow \mathfrak{sl}_2$-triple $\{e,h,f\}$, $h \in \mathfrak{h}$
$\alpha(h) \geq 0$ for $\alpha \in \Phi^+$

$\mathfrak{g}_i = \{X \in \mathfrak{g} \mid h.X = i \cdot X\}$

$\mathfrak{g}_0$ = reductive Lie algebra $\curvearrowright \mathfrak{g}_i$

G_0 = gp. $\leftrightsquigarrow \mathfrak{g}_0$ $\quad G_0$ Levi subgp

$P = P_{\geq 0} \leftrightsquigarrow \mathfrak{g}_{\geq 0}$ $\quad P$ parabolic subgp.

$\mathbb{C}^* \subset G$ coming from h, $\mathbb{C}^* \curvearrowright \mathcal{B}_e$

$Y_w := PwB \cap \mathcal{B}_e$ smooth

$$\begin{array}{c} Y_w \\ \downarrow \\ Y_w^{\mathbb{C}^*} \end{array} \quad \begin{array}{l} \\ \text{vector bundle, fiber } \mathbb{C}^m \\ \text{smooth} \end{array}$$

If $w \in W$ minimal length for W_0-coset (Weyl gp. of G_0) $\Rightarrow w^{-1}.\eta \cap \mathfrak{g}_2$ stable under $B \cap G_0$. (Borel for G_0)

Y_w nonempty $\iff w^{-1}\eta \cap \mathfrak{g}_2$ meets dense G_0-orbit in $\mathfrak{g}_2$ (i.e., $\mathcal{O}_e$)

Let $\mathcal{U}_2^{\text{dense}} = \{U \subseteq \mathfrak{g}_2 \mid B \cap G_0\text{-stable}, \ _{,B_0}\ U \cap \mathcal{O}_e^\circ \neq \phi\}$

$$X_U := \{ gB_0 \mid g \in G_0,\ g^{-1}.e \in U \}$$

$X_U \subseteq G_0/B_0$ is smooth projective

$Y_w := PwB \cap \mathcal{B}_e$ <u>smooth</u>

$\downarrow$ vector bundle.

$Y_w^{\mathbb{C}^*}$ <u>smooth</u> $\simeq X_U$ where $U \in \mathfrak{d}_2^{\text{dense}}$, $w^{-1}.\mathfrak{n} \cap \mathfrak{g}_2 = U$

Results from DLP paper:

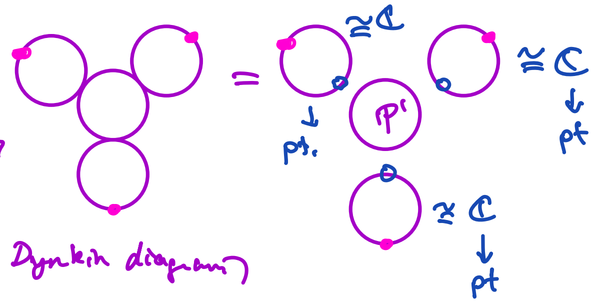
① $H^*(\mathcal{B}_{e_1}^{\mathbb{Z}}) \cong \bigoplus_{U \in \mathfrak{d}_2^{\text{dense}}} H^*(X_{U'}^{\mathbb{Z}}) \otimes V_U$ ← graded free $\mathbb{Z}$-mod., even degrees only

Now $A(e)$ also acts on X_U and this is an isomorphism as $A(e)$-modules (acting trivially on V_U).

② $H^{2i+1}(X_U) = 0$, $H^{2i}(X_U)$ has no torsion

<u>Conclusion</u>: $H^{2i+1}(\mathcal{B}_{e_r}^{\mathbb{Z}}) = 0$, $H^{2i}(\mathcal{B}_{e_r}^{\mathbb{Z}})$ has no torsion.

Subregular e in orbit in G_2: $\mathcal{B}e =$



4 $\mathbb{P}^1$'s

(D_4 Dynkin diagram)

$\boxed{\begin{array}{l} \text{one } X_U = 3 \text{ pts} \\ \text{one } X_U = \mathbb{P}^1 \end{array}}$

In E_8, there's a case with

$$420+756+1596+378+1092+168+70 = 4480 \quad Y_w$$

using $152+97+121+40+71+8+12 = 501 \quad X_U$

$A(e) \cong S_5$, 4480 is dim of irrep of $W(E_8) \to H^{top}(\mathcal{B}_e)^{S_5}$

Theorem (Precup-S; Xue)

For $I \in \mathcal{C}$ and $e \in \mathcal{N}$, as $A(e)$-modules,

$$H^*(\mathcal{B}_e^I) \cong \bigoplus_{U \in \mathcal{C}_2^{dense}} H^*(X_U) \otimes V_U^I$$

graded v. space even degrees only

Thus, $H^{2i+1}(\mathcal{B}_e^I) = 0$.

Remark: $\#\{w \mid w^{-1}.I \cap \mathfrak{g}_2 = U\} = \dim V_U^I$

minimal for W_0.

Lemma: Every X_U appears in the decomposition of $\mathcal{B}e$ (in DLP for e even)

Corollary: Every irreducible rep'n of $A(e)$ appearing in $H^*(X_U)$ appears in $H^*(\mathcal{B}e)$ (which then also appears in $H^{top}(\mathcal{B}e)$ by earlier work of Beynon-Spaltenstein, Shoji, and Lusztig)

<u>Corollary</u>: Every irreducible rep'n of $A(e)$ appearing in $H^*(\mathcal{B}_e^I)$ appears in the Springer correspondence, for all $I \in \mathcal{cl}$.

<u>Geometric Modular Law</u>

Name comes a combinatorial law for chromatic symmetric functions, Geometric version is in DLP for computing $H^*(X_U)$ in the exceptional groups.

<u>Definition</u>: A triple $I_1 \subset I_2 \subset I_3$ of elements of $\mathcal{cl}$ is called a modular triple if:

1. $\dim(I_3/I_1) = 2$
2. $\exists P$, a minimal parabolic subgroup of G, such that $P \cdot I_3 = I_3$, $P.I_1 = I_1$, and $P \cdot I_2 = I_3$.

possible to state using only the root system.

<u>Theorem (DLP)</u>

$$(q+1) H^*(X_{U_2}) = H^*(X_{U_3}) + q H^*(X_{U_1})$$

whenever $U_1 \subset U_2 \subset U_3$ is a modular triple in $\mathcal{cl}_2^{dense}$.

<u>Sketch</u>: $X_{U_2} \times \mathbb{P}^1 \cong$ Blow-up of X_{U_3} along X_{U_1}

Theorem: (Precup, S–)

$$(q+1)\,H^*(\mathcal{B}_e^{I_2}) = H^*(\mathcal{B}_e^{I_3}) + q\,H^*(\mathcal{B}_e^{I_1})$$

whenever $I_1 \subset I_2 \subset I_3$ is a modular triple in $\mathcal{L}$.

Sketch: Same proof, but use $H^{odd}(\mathcal{B}_e^I) = 0$ to get isomorphism of cohomology groups.

Remark: Isomorphism is $A(e)$-isomorphism.

Corollary: Geometric Modular Law implies the modular law of Guay-Paquet and Abreu-Negro in type A for chromatic symmetric fnctns attached to indifference graphs. (attached to Dyck paths)

Ideal	Min roots	$[1^7]$	$[2,1^5]$	$[3,1^4],\epsilon$	$[3,1^4],1$	$[3,2^2]$	$[3^2,1],\epsilon$	$[3^2,1],1$	$[5,1^2].\epsilon$	$[5,1^2],1$	$[7]$
20*	$\emptyset$	[2][4][6]									
19*	122	[2][4][6]	[2][2]								
18	112	[2][4][6]	[2][2][2]								
17	12	[2][4][6]	[2][2][3]								
16*	111	[2][4][6]	[2][2][2]	[2][2]	[2][2]						
15*	110	[2][4][6]	[2][2][3]	q[2]	[2][3]						
14*	100	[2][4][6]	[2][2][4]	0	[2][4]						
13*	111, 012	[2][4][6]	[2][2][3]	[2][2]	[2][2]	[2]					
12	011	[2][4][6]	[2][2][3]	[2][2][2]	[2][2][2]	[2][2]					
11	001	[2][4][6]	[2][2][3]	[2][2][3]	[2][2][3]	[2][3]					
10*	110, 012	[2][4][6]	$[2][2](1+q+2q^2)$	q[2]	[2][3]	[2]	1	1			
9	100, 012	[2][4][6]	$[2][2](1+q+2q^2+q^3)$	0	[2][4]	[2]	[2]	[2]			
8*	110, 011	[2][4][6]	$[2][2](1+q+2q^2)$	$2q+3q^2+q^3$	$[2](1+2q+2q^2)$	$[2](1+2q)$	0	[2]			
7*	010	[2][4][6]	$[2][2](1+q+2q^2+q^3)$	q[2][2]	[2][2][3]	[2][2][2]	0	[2][2]			
6*	100, 011	[2][4][6]	$[2][2](1+q+2q^2+q^3)$	q[2][2]	[2][2][3]	$[2](1+2q)$	q	$1+2q$	1	1	
5	110, 001	[2][4][6]	$[2][2](1+q+2q^2)$	$q[2](2+2q+q^2)$	$[2](1+2q+3q^2+q^3)$	[2][2][2]	0	[2]	1	1	
4	100, 001	[2][4][6]	$[2][2](1+q+2q^2+q^3)$	q[2][2][2]	$[2](1+2q+3q^2+2q^3)$	$[2](1+2q+2q^2)$	0	[2][2]	[2]	[2]	
3	010, 001		$[2][2](1+q+2q^2+q^3)$	q[2][2][2]	$[2](1+2q+3q^2+2q^3)$	$[2](1+2q+2q^2)$	0	[2][2]	[2]	[2]	
2*	100, 010	[2][4][6]	$[2][2](1+q+2q^2+2q^3)$	q^2[2]	$[2](1+2q+2q^2+2q^3)$	[2][2][2]	q[2]	$1+3q+2q^2$	0	[2]	
1*	Δ	[2][4][6]	$[2][2](1+q+2q^2+2q^3)$	$2q^2+3q^3+q^4$	$1+3q+5q^2+6q^3+3q^4$	$1+3q+5q^2+3q^3$	q^2	$1+3q+3q^2$	$2q$	$1+3q$	1

TABLE 1. The polynomials $\mathsf{P}(\mathcal{B}_x^I;\chi)$ for B_3

$[m] = [m]_q = \dfrac{q^m - 1}{q-1}$

$\chi \in \mathrm{Irrep}(A(e))$

$x = e$

Poincare polynomials (even degrees)

$\pi_I : G \times^B I \to \mathcal{N}$ $\quad$ G-equivariant

$\underline{\mathbb{C}}$ constant sheaf on $G \times^B I$,

$\underline{\mathbb{C}}[\dim(G \times^B I)]$ is perverse, as $G \times^B I$ smooth

$$R\pi_{I*}\left(\underline{\mathbb{C}}[\dim(G\times^B I)]\right) \cong \bigoplus_{(\mathcal{O},\mathcal{L})} IC(\mathcal{O},\mathcal{L}) \otimes V^I_{\mathcal{O},\mathcal{L}}$$

(by decomposition theorem) $\qquad$ $\uparrow$ graded v. space

$\mathcal{O}$ nilp. orbit, $\mathcal{L}$ local system on $\mathcal{O}$, and $IC(\mathcal{O},\mathcal{L})$ intersection cohomology

$$\boxed{\begin{array}{l}\text{Back to case of } I=\mathfrak{n} \Rightarrow \text{Springer correspondence} \\ (\mathcal{O},\mathcal{L}) \text{ appearing on RHS for } I=\mathfrak{n} \xleftrightarrow{1-1} \text{Irred. reps. of } W \\ \qquad (\mathcal{O},\mathcal{L}) \rightsquigarrow U_{(\mathcal{O},\mathcal{L})}\end{array}}$$

More generally,

<u>Theorem (Precup, S-)</u>:

For $I \in \mathcal{U}$, if $(\mathcal{O},\mathcal{L})$ appears on RHS, then $(\mathcal{O},\mathcal{L})$ appears on RHS for $I=\mathfrak{n}$. That is, the irreducible local system $\mathcal{L}$ on $\mathcal{O}=\mathcal{O}_e$ arises from a $\chi \in \mathrm{Irrep}(A(e))$ that appears in the Springer correspondence (i.e., in $H^{top}(\mathcal{B}_e)$)

<u>Sketch</u>: Use proper change to translate to $A(e)$-rep'n on $H^*(\mathcal{B}_e^I)$.

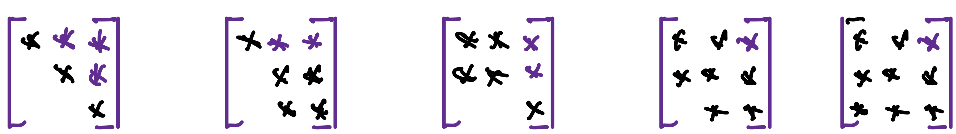
<u>Remark</u>: This was a conjecture of Brosnan

Ideal	Min roots	$\emptyset, [1^3]$	$\emptyset, [2,1]$	$[1^3], \emptyset$	$[1], [1^2]$	$[1^2], [1]$	$\emptyset, [3]$	$[1], [2]$	$[2,1], \emptyset$	$[2], [1]$	$[3], \emptyset$
20	$\emptyset$	$[2][4][6]$									
19	122	$[2][4][5]$	$q^3[2][2]$								
18	112	$[2][4][4]$	$q^2[2][2][2]$								
17	12	$[2][3][4]$	$q[2][2][3]$								
16	111	$[2][3][4]$	$q^2[2][2]$	$q^2[2][2]$	$q^2[2][2]$						
15	110	$[2][2][4]$	$q[2][3]$	$q^2[2]$	$q[2][3]$						
14	100	$[2][4]$	$[2][4]$	0	$[2][4]$						
13	111, 012	$[2][3][3]$	$q[2][2][2]$	$q^2[2]$	$q^2[2]$	$q^2[2]$					
12	011	$[2][2][3]$	$q[2][2]$	$q[2][2]$	$q[2][2]$	$q[2][2]$					
11	001	$[2][3]$	0	$[2][3]$	$[2][3]$	$[2][3]$					
10	110, 012	$[2][2][3]$	$2q[2][2]$	q^2	$q[2][2]$	q^2	q^2	q^2			
9	100, 012	$[2][3]$	$[2][2][2]$	0	$[2][3]$	0	$q[2]$	$q[2]$			
8	110, 011	$[2][2][2]$	$2q[2]$	$q[2]$	$2q[2]$	$2q[2]$	0	$q[2]$			
7	010	$[2][2]$	$[2][2]$	0	$[2][2]$	$[2][2]$	0	$[2][2]$			
6	100, 011	$[2][2]$	$1+3q+q^2$	q	$1+3q+q^2$	$2q$	q	$2q$	q	q	
5	110, 001	$[2][2]$	q	$[2][2]$	$1+3q+q^2$	$1+3q+q^2$	0	q	q	q	
4	100, 001	$[2]$	$[2]$	$[2]$	$2[2]$	$2[2]$	0	$[2]$	$[2]$	$[2]$	
3	010, 001	$[2]$	$[2]$	$[2]$	$2[2]$	$2[2]$	0	$[2]$	$[2]$	$[2]$	
2	100, 010	$[2]$	$2[2]$	0	$2[2]$	$[2]$	$[2]$	$2[2]$	0	$[2]$	
1	Δ	1	2	1	3	3	1	2	2	3	1

TABLE 2. The polynomials f^I_φ for B_3

these are polynomials for graded v. space $V^I_{\sigma,\mathcal{L}}$ indexed by $\mathcal{U}_{\sigma,\mathcal{L}} \in \mathrm{Irrep}(W)$

Repeat the story for $H = I^\perp$ ← under Killing form



$\mathfrak{b}$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\mathfrak{g}$

$\pi_H : G \times^{B} H \to \mathfrak{g}$

Thm: $R\pi_{H*}(\underline{\mathbb{C}}[-]) \cong \bigoplus_{\substack{\mathcal{U} \in \mathrm{Irr}(W) \\ \mathcal{U} = \mathcal{U}_{(\sigma,\mathcal{L})}}} IC(\mathfrak{g}, \mathcal{L}_{\mathcal{U}}) \otimes V^I_{(\sigma,\mathcal{L})^\vee}$ ← same graded v. spaces

(conjecture of Brosnan)

$\mathcal{U}_{(\sigma,\mathcal{L})^\vee} = \mathcal{U} \otimes \mathrm{sgn}$

Consider $\pi_H^{-1}(s)$, $s \in \mathfrak{g}$ regular semisimple. It's cohom'l. carries rep'n of W (defined by Brosnan-Chow) and it agrees with Tymoczko's dot action.

$\boxed{\text{Shareshian-Wachs Conjecture}}$ Proved by Brosnan-Chow & Guay-Paquet

The representation of S_n corresponding to $X_{G_D}(\vec{x}, q)$ is the S_n rep'n on $H^*(\pi_H^{-1}(s)) \otimes \mathrm{sgn}$.

$$\left(\underset{I^\perp}{\overset{H}{\shortparallel}} \longleftrightarrow I \longleftrightarrow \underset{D}{\text{Dyck path}} \right)$$

Finally, graded v. spaces $V^\perp_{(\sigma,\mathcal{L})}$ controls the rep'n of W on $H^*(\pi_H^{-1}(s))$. That is, $\mathrm{Hom}_W(\mathcal{U}_{(\sigma,\mathcal{L})}, H^*(\pi_H^{-1}(s))) \cong V^I_{(\sigma,\mathcal{L})^\vee}$