Lie groups seminar

Calogero-Moser spaces vs unipotent representations

MIT

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$$W = \langle \mathrm{Ref}(W) \rangle \subset GL_{\mathbb{C}}(V) \quad \text{finite}$$
$$\mathrm{Ref}(W) = \{ s \in W \mid \operatorname{codim} V^s = 1 \}$$

W rational + diagram auto

parameter k

$\rightsquigarrow$ $Z_k(W)$ CM space

- Affine Poisson variety
- $\mathbb{C}^\times$ - action

? ? ?

$\longleftrightarrow$

Similar combinatoric

W rational, $k = k_{eq}$

G reductive group $/\mathbb{F}_q$

$G = \mathbb{G}(\mathbb{F}_q)$

Lusztig: $\mathrm{Unip}(G) \subset \mathrm{Irr}(G)$

Unipotent representations

Examples.

① Gordon - Martino conjecture (2006)

W real, k real $\Rightarrow$ $Z_k^{\mathbb{C}^\times} \overset{\sim}{\longleftrightarrow} \left\{ \begin{matrix} \text{Two-sided} \\ k\text{-cells} \\ \text{of } W \end{matrix} \right\}$

Lusztig '80's

$\overset{\sim}{\longleftrightarrow}$ W rational, $k = k_{eq}$, G split

$\left\{ \begin{matrix} \text{Lusztig families} \\ \text{of } \mathrm{Unip}(G^F) \end{matrix} \right\}$

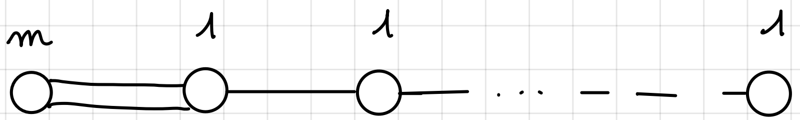
(true for W classical, H_3, F_4)

Gordon - Martino '06

Thiel - B. '13 '21

② Type B_n/C_n $\quad G = SO_{2n+1}(q)/Sp_{2n}(q)$

$\mathbf{k}_m \rightsquigarrow$



$m = 1 \Longrightarrow \mathbf{k}_{eq} = \mathbf{k}_1$

$W = \langle Ref(W) \rangle$ $\quad$ W rational

param. $\mathbf{k}$ $\qquad\qquad$ $\mathbb{G}$ reductive$/\mathbb{F}_q$

$Z_{\mathbf{k}}(W)$ CM space $\quad \overset{???}{\underset{\mathbf{k}=\mathbf{k}_{eq}}{\longleftrightarrow}} \quad$ $G = \mathbb{G}(\mathbb{F}_q)$

- Affine Poisson $\qquad$ $Unip(G) \subset Irr(G)$
- $\mathbb{C}^\times$-action $\qquad$ unipotent

$$\left\{ \begin{matrix} \text{Symplectic} \\ \text{leaves of } Z_{\mathbf{k}_m} \end{matrix} \right\} \overset{\sim}{\underset{m \in \mathbb{Z}_{\geq 0}}{\longleftrightarrow}} \left\{ r \geq 0 \;\middle|\; r^2 + mr \leq n \right\} \overset{\sim}{\underset{m=1}{\longleftrightarrow}} \left\{ \begin{matrix} \text{Harish-Chandra} \\ \text{series in } Unip(G) \end{matrix} \right\}$$

Bellamy-Thiel '17 $\qquad$ Lusztig '78

③ $W = \mathfrak{S}_n$, $\quad G = GL_n(q)$, $\quad \ell$ prime $\nmid q$, $\quad d = o(q \bmod \ell)$

$$\left\{ \begin{matrix} \text{Irr. components} \\ \text{of } Z_{\mathbf{k}_{eq}}^{\mu_d} \end{matrix} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{matrix} d\text{-cores of} \\ \text{partitions of } n \end{matrix} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{matrix} \ell\text{-blocks} \\ \text{in } Unip(G) \end{matrix} \right\}$$

Haiman '02 $\qquad$ Fong-Srinivasan '82

Gordon '08

Maksimau-B.-'21

- $\dim_{\mathbb{C}} V < \infty$
- $W \subset GL(V)$ finite
- $\mathrm{Ref}(W) = \{ s \in W \mid \operatorname{codim} V^s = 1 \}$
- $\mathcal{A} = \{ V^s \mid s \in \mathrm{Ref}(W) \}$
- $V^{reg} = V \setminus \bigcup_{H \in \mathcal{A}} H \underset{\text{(Steinberg)}}{=} \{ v \in V \mid \mathrm{Stab}_W(v) = 1 \}$
- $\tau \in N_{GL(V)}(W)$ of finite order

$$\boxed{W = \langle \mathrm{Ref}(W) \rangle}$$

Def. . τ is called $(W\text{-})$full if $\dim V^\tau = \max_{w \in W} \dim V^{w\tau}$

. τ is called $(W\text{-})$regular if $V^\tau \cap V^{reg} \neq \emptyset$

Springer (1974). * τ is full $\iff V^\tau \longrightarrow (V/W)^\tau$ is onto

* regular $\implies$ full

- $W_\tau = \mathrm{Stab}^{set}_W(V^\tau) / \mathrm{Stab}^{pt}_W(V^\tau)$

Theorem (Springer '74: regular; Lehrer–Springer '99: full)

(a) W_τ is a reflection group on V^τ.

(b) $V^\tau / W_\tau \xrightarrow{\sim} (V/W)^\tau$

Remark. τ regular $\implies W_\tau = W^\tau$.

$W = \langle \mathrm{Ref}(W) \rangle$ — W rational

param. k

$Z_k(W)$ CM space

. Affine Poisson

. $\mathbb{C}^\times$-action

???

$\longleftrightarrow$

$k = k_{eq}$

$\mathbb{G}$ reductive$/\mathbb{F}_q$

$G = \mathbb{G}(\mathbb{F}_q)$

$\mathrm{Unip}(G) \subset \mathrm{Irr}(G)$

unipotent

From now on, τ is full

Definition. A parabolic subgroup P of W is called τ-split if $P = \mathrm{Stab}_W(v)$ for some $v \in V^\tau$.

Corollary

$$\left\{ \begin{matrix} \tau\text{-split parabolic} \\ \text{subgroups of } W \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{parabolic} \\ \text{subgroups of } W_\tau \end{matrix} \right\}$$

$$P \longmapsto P_\tau$$

Calogero-Moser spaces (Etingof-Ginzburg '02)

- $H \in \mathfrak{A} \rightsquigarrow W_H = \mathrm{Stab}^{pt}_W(H) \rightsquigarrow e_H = |W_H|$
- $\nabla(V,W) = \{(\Omega, j) \mid \Omega \in \mathfrak{A}/W$ and $0 \leq j \leq e_\Omega - 1\}$
- Fix $k = (k_{\Omega,j})_{(\Omega,j) \in \nabla(V,W)} \in \mathbb{C}^{\nabla(V,W)}$

Cherednik algebra:

$$H_{t,k} = \frac{T(V \times V^*) \rtimes W}{\left\langle \begin{array}{l} \forall x, x' \in V^*, \forall y, y' \in V \\ (a)\ [x,x'] = [y,y'] = 0 \\ (b)\ [y,x] = t\langle y,x\rangle \\ \quad + \sum_{H,j} (k_{H,j+1} - k_{H,j}) \frac{\langle y, \alpha_H\rangle\langle \alpha_H^\vee, x\rangle}{\langle \alpha_H^\vee, \alpha_H\rangle} \varepsilon_{H,j} \end{array} \right\rangle}$$

where $H = \operatorname{Ker} \alpha_H$, $V = H \oplus \mathbb{C}\alpha_H^\vee$

$$\varepsilon_{H,j} = \frac{1}{e_H} \sum_{w \in W_H} \det(w)^{-j} w \in \mathbb{C}W_H.$$

Definition. Calogero-Moser space

$$Z_k = \operatorname{Spec} Z(H_{0,k})$$

(irreducible, normal, $\mathbb{C}^\times$-action, Poisson)

Finite reductive groups

- (W, τ) rational (i.e. $\exists V_{\mathbb{Q}} \subset V$ such that $V = \mathbb{C} \otimes_{\mathbb{Q}} V_{\mathbb{Q}}$ and $V_{\mathbb{Q}}$ is $W\langle\tau\rangle$-stable
- p prime, $q = p^m$

$\Rightarrow$ $\mathbb{G}$ reductive group over $\overline{\mathbb{F}}_q = \mathbb{F}$

$+$ $F : \mathbb{G} \longrightarrow \mathbb{G}$

$+$ F-stable maximal torus $\mathbb{T}$ $\subset$ F-stable Borel $\mathbb{B}$

such that $W = N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$

- $V_{\mathbb{Q}} \underset{W,\, \tau = q^{-1}F}{\simeq} \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Hom}(\mathbb{T}, \mathbb{F}^\times)$

- Finite reductive group $G = \mathbb{G}^F$

Lusztig. $\operatorname{Unip}(G) \subset \operatorname{Irr}(G)$

Unipotent representations appearing in $H^i_c(\mathbb{G}\text{-orbit} \cap \text{Frobenius graph in } \mathbb{G}/\mathbb{B} \times \mathbb{G}/\mathbb{B}, \overline{\mathbb{Q}}_\ell)$

. Assume that $\tau(k) = k$. Then τ acts on Z_k

. Z_k^τ is Poisson $\rightsquigarrow$ symplectic leaves
Brown - Gordon

Fact (Brown - Gordon, B.)

$$\#\{\text{symplectic leaves}\} < \infty$$

. A point $p \in Z_k^\tau$ is called τ-cuspidal if it is a symplectic leaf.

Theorem (Bellamy, Losev ?? for $\tau = \mathrm{Id}_V$; B. '21 in general)

$$\left\{ \begin{matrix} \text{symplectic} \\ \text{leaves in } Z_k^\tau \end{matrix} \right\} \xleftrightarrow{\sim} \left\{ \begin{matrix} (P, p) \text{ where } P \\ \text{is } \tau\text{-split and} \\ p \text{ is } \tau\text{-cuspidal} \\ \text{in } Z_k(V/V^P, P)^\tau \end{matrix} \right\}$$

$$\mathcal{S}_{P,\lambda} \longleftarrow (P, \lambda)$$

$\dim = 2 \dim (V^P)^\tau$

. Fix $d \geq 1$, $\mu_d = \langle \zeta_d \rangle$

. Fix $w \in W$ such that $\dim V^{\zeta_d w \tau}$ is maximal: then $\tau_d = \zeta_d w \tau$ is full.

. An F-stable Levi subgroup is called d-split if $(\mathbb{L}, F) \leftrightarrow (P, w\tau)$ where P is a τ_d-split parabolic subgroup of W.

. $\rho \in \mathrm{Unip}(\mathbb{G})$ is called d-cuspidal if $\deg \rho$ is divisible by $\Phi_d(q)^{\alpha_d}$ where α_d is the valuation of Φ_d in the polynomial order of $\mathbb{G}_{ad}$

Theorem (Broué - Malle - Michel '93)

(a) $\mathrm{Unip}(G) = \bigsqcup_{(\mathbb{L}, \lambda)} \mathrm{Unip}(G, \mathbb{L}, \lambda)$

d-split $(\mathbb{L}, \lambda)$ d-cuspidal

(b) $N_G(\mathbb{L}, \lambda)/_L$ is a reflection group on $(V^P)^{\tau_d}$ if $\mathbb{L} \leftrightarrows (P, w\tau)$

(c) $\exists$ $\mathrm{Irr}\, N_G(\mathbb{L}, \lambda)/_L \xleftrightarrow{\sim} \mathrm{Unip}(G, \mathbb{L}, \lambda)$

("d - Harish - Chandra theory")

Conjecture (B. '21). Fix (P, z) as above. Then

$$\mathcal{F}_{P,z}^{\mathrm{mon}} \underset{\mathbb{C}^\times,\ \mathrm{Poisson}}{\simeq} \mathcal{Z}_{k'_{P,z}}\left(\left(V^P\right)^\tau, N_{W_z}(P_z)/P_z\right)$$

for some parameter $k'_{P,z}$.

Question. $k'_{P,z}$?

Conjecture (BMM '93). Fix $(\mathbb{L}, \lambda)$ as above. Then there exists an explicit Deligne-Lusztig variety Y such that:

(a) $\mathrm{Unip}(G, \mathbb{L}, \lambda)$ is the set of components of $\bigoplus_{i \geq 0} H_c^i(Y) \otimes_{L^F} \lambda =: R_L^G \lambda$

(b) $\mathrm{End}_G R_L^G(\lambda) = \mathrm{End}_G^{gr}(R_L^G(\lambda)) \simeq \mathcal{H}ecke\left(N_G(\mathbb{L}, \lambda)/L, k_{L,\lambda}\right)$

for some explicit parameter $k_{L,\lambda}$.

Conjecture (B. '21). Fix (L, λ) as in BMM and assume that $N_G(\mathbb{L}, \lambda) = N_G(\mathbb{L})$ (99.9 % of the cases...). Let $P = \mathrm{Weyl}(\mathbb{L})$: it is τ-split.

Let z be the point of $\left(\mathcal{Z}_{k_{eq}}\left(V/V^P, P\right)^\tau\right)^{\mathbb{C}^\times}$ associated with the family of λ through Gordon–Martino conjecture. Then:

(a) z is τ-cuspidal.

(b) $k'_{P,z} = k_{L,\lambda}$ (makes sense since $N_{W_z}(P_z)/P_z = N_G(\mathbb{L})/L$)

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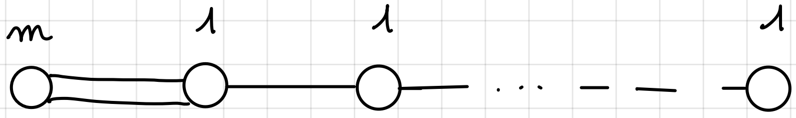
$k_m \rightsquigarrow$



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param. k $\quad$??? $\quad$ $\mathbb{G}$ reductive$/\mathbb{F}_q$

$Z_k(W)$ CM space $\xleftrightarrow[k = k_{eq}]{???}$ $G = \mathbb{G}(\mathbb{F}_q)$

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Comments.

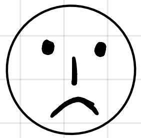
(1) **Riche - Williamson.** Proof of the linkage principle through a bijection

$$\{\text{Irr. comp. of } \mathcal{G}r^{\mu_p}_{\mathrm{aff}}\} \xleftrightarrow{\ \sim\ } \{\text{blocks of } \mathrm{Rep}(G)\}$$

(2) It seems that one can recover, from the geometry of $Z_{k_{eq}}$, the following data:

* Lusztig families
* Harish-Chandra series
* Parameters of endomorphism algebras

\+ many compatibilities.

But we do not recover the unipotent characters . Where are they hidden?

(3) Lusztig (Coxeter), BMM (some complex reflection groups)

$\Longrightarrow$ $\mathrm{Unip}(W)$ + data

For $I_2(m)$ or G_4, this fits perfectly with the above story.