Harish-Chandra modules over quantizations of nilpotent orbits.

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1) Harish-Chandra modules.
2) Restriction functors.
3) Classification results & ideas of proof.

1) General setting: $\mathcal{A}$ assoc've algebra$/\mathbb{C}$, K is an algebraic group acting rationally on $\mathcal{A}$ w. a quantum comoment map, i.e. $\Phi: \mathfrak{k} \longrightarrow \mathcal{A}$ that is K-equivariant & satisfies $[\Phi(\xi), \cdot] = \xi_{\mathcal{A}}$, $\forall \xi \in \mathfrak{k}$, where $\xi_{\mathcal{A}}$ is the derivation of $\mathcal{A}$ coming from $K \curvearrowright \mathcal{A}$.

Ex: G is a semisimple alg'c group, $K \to G$

$\mathcal{A} = U(\mathfrak{g})$, $\Phi: \mathfrak{k} \to \mathfrak{g} \subset U(\mathfrak{g})$.

Def'n: • Let $\lambda \in (\mathfrak{k}^*)^K$. By a (K,λ)-equivariant $\mathcal{A}$-module we mean an $\mathcal{A}$-module $\mathcal{M}$ equipped with a rational representation of K s.t. $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{M} \to \mathcal{M}$ is K-equivariant &

$$\xi_{\mathcal{M}} m = \varphi(\xi) m - \langle \lambda, \xi \rangle m \quad \forall \xi \in \mathfrak{k},\ m \in \mathcal{M},$$

When $\lambda = 0$ we omit it from the notation.

• Suppose $K \to G$ gives an isomor'm $\mathfrak{k} \xrightarrow{\sim} \mathfrak{g}^{\sigma}$ for some involution σ. A Harish-Chandra $(U(\mathfrak{g}), K)$-module is a finitely generated K-equivariant $U(\mathfrak{g})$-module.

Remark: if $K \subset G$, then HC $(\mathfrak{g}, K)$-modules are related to representations of the real form $G_{\mathbb{R}}$ corresponding to σ. In general, they are related to representations of nonlinear covers of $G_{\mathbb{R}}$.

Question: Given a primitive ideal $I \subset \mathcal{U}(\mathfrak{g})$ (= annihilator of a simple module) classify irreducible HC modules M w. $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(M) = I$.

A very interesting class of primitive ideals are unipotent ones (I.L., Mason-Brown, Matvieievskyi '21). These ideals are constructed from equivariant covers of nilpotent orbits. We'll concentrate on the ideals arising from nilpotent orbits $\mathbb{O} \subset \mathfrak{g}^*$ s.t.

$$\mathrm{codim}_{\overline{\mathbb{O}}} (\overline{\mathbb{O}} \setminus \mathbb{O}) \geqslant 4$$

For example, all "rigid" orbits but six in exceptional algebras satisfy this condition.

The ideal $I = I_{\mathbb{O}}$ arising from $\mathbb{O}$ has the following properties:

(1) I is maximal.

(2) $\mathcal{A} := \mathcal{U}(\mathfrak{g})/I$ admits a filtration w. $\operatorname{gr}\mathcal{A} \xrightarrow{\sim} \mathbb{C}[\mathbb{O}]$ (isom'm of graded Poisson algebras)

Rem: Filtered algebras satisfying (2) are naturally classified by $H^2(\mathbb{O},\mathbb{C})$. We pick one corresponding to 0, the canonical quantization. If $\mathbb{O}$ is (birationally) rigid, then $H^2(\mathbb{O},\mathbb{C}) = \{0\}$, so I is uniquely determined by condition (2). In fact, our approach works for all quantizations of $\mathbb{C}[\mathbb{O}]$ under the codim condition - which is crucial.

Examples:

1) $K = SL_2 \twoheadrightarrow SO_3 \hookrightarrow SL_3 = G$, $\mathbb{O}$ is minimal nilpotent orbit, Jordan type $(2,1)$, $\dim = 4$.

$\mathcal{A} = \mathcal{D}^{-3/2}(\mathbb{P}^2)$ (global twisted differential operators in $\frac{1}{2}\cdot$ canonical class). Using localization thm we see that there are three HC $(\mathcal{A}, K)$-modules. Two are SO_3-equivariant (they come from the open K-orbit in $\mathbb{P}^2$), one is not - it comes from the closed orbit.

2) $G = Sp_4$, $\mathfrak{k} = \mathfrak{gl}_2$, $\mathbb{O}$ = minimal orbit, Jordan type $(2,2)$, $\dim = 4$. In fact, $\overline{\mathbb{O}} = \mathbb{C}^4/\{\pm 1\}$. The only quantization is $\mathcal{A} = W(\mathbb{C}^4)^{\{\pm 1\}}$, where W is for the Weyl algebra. For a 2-fold cover K of $GL(2)$, there are 4 irreducible HC modules: the $\{\pm 1\}$-isotypic components in quantizations of $GL(2)$-equivariant lagrangian subspaces in $\mathbb{C}^4$. All HC modules factor through this cover (name: metaplectic modules)

2) Restriction functors.

$\operatorname{codim}_{\overline{\mathbb{O}}}(\overline{\mathbb{O}}\setminus\mathbb{O})\geqslant 4 \Rightarrow \forall$ K-orbit $\mathbb{O}_K\subset\mathbb{O}\cap\mathfrak{k}^\perp$ have $\operatorname{codim}_{\overline{\mathbb{O}}_K}(\overline{\mathbb{O}}_K\setminus\mathbb{O}_K)\geqslant 2$

Fact (Vogan): The associated variety of any irreducible HC $(\mathcal{A},K)$-module is the closure of a single K-orbit in $\mathbb{O}\cap\mathfrak{k}^\perp$.

So fix $\mathbb{O}_K\subset\mathbb{O}$ & pick $\chi\in\mathbb{O}_K \rightsquigarrow$ stabilizer $K_\chi\subset K$ $\rightsquigarrow$ its reductive part R.

Canonical bundle of $\mathbb{O}_K\longleftrightarrow$ character of R

$\kappa:=\frac{1}{2}$ this character $\in\mathfrak{r}^{*R}$

Def'n: • $(R,\kappa)\text{-mod}=\{$fin. dim R-modules, where $\mathfrak{r}$ acts by $\kappa\}$ - semisimple category.

$HC_{\mathbb{O}_K}(\mathcal{A},K)=\{HC\ (\mathcal{A},K)\text{-modules whose associated variety is } \overline{\mathbb{O}}_K\}$.

I.L. (12) $\rightsquigarrow$ restriction functor $\bullet_\dagger$: $HC(\mathcal{U}(\mathfrak{g}), K) \longrightarrow$ "$HC(\mathcal{W}, R)$", where $\mathcal{W}$ is the W-algebra quantizing Slodowy slice to $\mathbb{O}$; $I \rightsquigarrow$ codim 1 ideal in $\mathcal{W}$ which acts by 0 on $M_\dagger$ for $M \in HC(\mathcal{A}, K)$. So $\bullet_\dagger$ restricts to

$$HC_{\mathbb{O}_K}(\mathcal{A}, K) \longrightarrow (R, \kappa)\text{-mod}$$

Properties: 1) $\bullet_\dagger$ is a fully faithful embedding.

2) if $\operatorname{codim}_{\overline{\mathbb{O}}_K} \overline{\mathbb{O}}_K \setminus \mathbb{O} \geqslant 3$, then $\bullet_\dagger$ is equivalence.

This is proved in I.L.'s papers (08, 15) for bimodules, the general case is analogous. Another proof of 2) was found by Leung-Yu.

Example: 1) $\mathbb{O} = \mathbb{O}_{min} \subset \mathfrak{sl}_3$, $K = SL_2$,

$\mathbb{O} \cap \mathfrak{k}^\perp$ is single K-orbit $(\mathbb{C}^2 \setminus \{0\}) / \{\pm 1, \pm\sqrt{-1}\}$

(R, κ)-mod $\simeq (\mathbb{Z}/4\mathbb{Z})$-mod; SO_3-equivariant irreducile HC modules $\mapsto$ irreps where $\pm\sqrt{-1}$ act trivially; non-SO_3-equivariant module goes to one of remaining two. So $\bullet_\dagger$ is NOT equivalence

2) $\mathbb{O} = \mathbb{O}_{min} \subset \mathfrak{sp}_4$, $\mathfrak{k} = \mathfrak{gl}_2$: $\mathbb{O} \cap \mathfrak{k}^\perp = \sqcup$ 2 orbits both $\simeq (\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$ $(\subset (\mathbb{C}^4 \setminus \{0\})/\{\pm 1\} = \mathbb{O})$

So for each orbit (R, κ)-mod $\simeq \mathbb{Z}/2\mathbb{Z}$-mod.

And $\bullet_\dagger$ is an equivalence.

Rem: Here's a way to compute $\bullet_\dagger$. Take a good filtration on $M \rightsquigarrow \operatorname{gr} M|_{\mathbb{O}_K}$. By a result of Vogan, this is a twisted local system w. half-canonical twist. Its fiber at χ is an object in (R, κ)-mod. This fiber is isomorphic to $M_\dagger$.

3) Classification result & ideas of proof.

Recall that $\operatorname{codim}_{\bar{\mathbb{O}}} \bar{\mathbb{O}} \setminus \mathbb{O} \geq 4$.

Thm: The functor $\bullet_\dagger : HC_{\mathbb{O}_K}(\mathfrak{g}, K) \to (R, \mathfrak{k})\text{-mod}$ is an equivalence if $K \subset G$ or $\mathfrak{g}$ is of type B, C, D.

There's also a description of the image for $\mathfrak{g} = \mathfrak{sl}_n$, $K = Spin_n$. Here under $\operatorname{codim}_{\bar{\mathbb{O}}} \bar{\mathbb{O}} \setminus \mathbb{O} \geq 4$ $\mathfrak{p}^{*R} = \{0\}$ so $(R, \mathfrak{k})\text{-mod} = K_Q / K_Q^\circ\text{-mod}$.

Thm: There are elements $a_1, \dots a_k \in K_Q / K_Q^\circ$ (described explicitly up to an isomorphism) such as TFAE:

- $V \in K_Q / K_Q^\circ\text{-mod}$ is in the image of $\bullet_\dagger$
- $-\sqrt{-1}$ is not an eigenvalue of a_i in V, $\forall i$.

A general idea of proof is as follows

Let $\mathbb{O}' \subset \overline{\mathbb{O}}$ be a codim 4 orbit w. $\mathbb{O}' \cap \overline{\mathbb{O}}_K = \emptyset$

Pick $e' \in \mathbb{O}' \cap \overline{\mathbb{O}}_K \rightsquigarrow \mathfrak{sl}_2$-triple (e', h', f') w. $e', f' \in \mathfrak{k}^{\perp}$

$\rightsquigarrow$ slice $S' := (e' + \mathfrak{z}_{\mathfrak{g}}(f')) \cap \overline{\mathbb{O}} \curvearrowleft Q' := Z_G(e', h', f')$

A description of S' is known in all cases (Kraft-Procesi for classical types, Fu-Juteau-Levy-Sommers for exceptional types), it's as follows:

- Type A: $S' = \overline{\mathbb{O}}_{min} \subset \mathfrak{sl}_3$, Q'-action factors through PGL_3.

Types BCD: $S' = \overline{\mathbb{O}}_{min} \subset \mathfrak{sp}_4$, Q'-action factors through SO_5.

Exceptional types: these two + variations & 3 exceptional types.

Now take $V \in (R, \mathfrak{k})$-mod $\rightsquigarrow$ twisted local system $\mathcal{L}_V$ on $\mathbb{O}_K$. The main observation is the following two implications:

- Easy: if $V = M_+$, then $\mathcal{L}_V|_{S' \setminus \{0\}}$

coincides w. $\mathrm{gr}(\mathcal{M}|_{S'})|_{S'\setminus\{0\}}$, where $\bullet|_{S'}$: $HC(\mathcal{A}, K) \longrightarrow HC(\mathcal{A}_{S'}, K_{Q'})$, where $\mathcal{A}_{S'}$ is the canonical quantization of $\mathbb{C}[S']$ and $K_{Q'} = K \cap Q'$. The category $HC(\mathcal{A}_{S'}, K_{Q'})$ makes sense when $S' = \overline{\mathbb{O}}_{min}$; it needs to be modified for exceptional 4-dim'l singularities.

• Harder: Suppose that $\forall\, \mathbb{O}'\ \exists$ $\mathcal{M}_{S'} \in HC(\mathcal{A}_{S'}, K_{Q'})$ w. $\mathrm{gr}\,\mathcal{M}_{S'} = \Gamma(\mathcal{L}_V|_{S'\setminus\{0\}})$. Then $V \in \mathrm{im}(\bullet_+)$.