Topic: representation theory of a reductive group $G$ over a local or finite field $F$.

Fundamental example: $G=GL(n)$, $n \times n$ matrices.

A parabolic subgroup $P$ of $G$ is one with $G/P$ projective. Example: Grassmann variety $X = m$-diml subspaces of $n$-diml vec space = $GL(n)/P_{\{m,n-m\}}$.

Parabolic subgroups are important because ANY $G/H$ can be deformed to $G/Hd$ with $Hd$ normal in parabolic $P$.

Original motivation: big idea from 1950s was that many representations are parabolically induced: sections of vector bundles over $G(F)/P(F)$.

Langlands idea from 1960s: all reps of $G(F)$ indexed by $f: W'F \rightarrow LG$; here $W'F$ is the Weil-Deligne group of $F$, and $LG = V_G \rtimes G$; $V_G$ is cplx dual group, $G$ is Gal($F$).

MORE PRECISELY: $G//F =$ all $F$-forms inner to $G(F) \iff$ action of $G$ on based root datum $(P, X^*, P^v, X^*)$

\[
\begin{align*}
&\iff \text{action of } G \text{ on dual based root datum } (P^v, X^*, P, X^*) \\
&\iff \text{action of } G \text{ on complex dual group } V_G \\
&\iff \text{Langlands L-group } LG = V_G \rtimes G
\end{align*}
\]

$V_G$ and a covering group $V_G^{\text{Can}}$ act on Langlands parameters $f \rightarrow$ stabilizer $S^{\text{Can}}(f)$, component group $A^{\text{Can}}(f)$.

COMPLETE LANGLANDS PARAMETER is pair $(f, x)$ with $x$ irreducible of $A^{\text{Can}}(f)$.

LOCAL LANGLANDS CONJECTURE: pairs ("rigid inner twist" $G(F)$, irr $p$ of $G(F)$)/$G$ conjugacy $\iff$ pairs $(f, x)/V_G^{\text{Can}}$ conjugacy

This is proven by Adams-Barbasch-V in 1992 for $F=R$, and formulated by Tasho Kaletha for $F$ characteristic zero $p$-adic.
Details A: p-adic case

\[ \mathbb{F} \text{ p-adic } [\text{char } 0], \quad \Gamma = \text{Gal}(\overline{\mathbb{F}}/\mathbb{F}) \]

\[ \mathcal{O}_F / \mathcal{P}_F = \mathbb{F}_q \]

\[ \| \cdot \| : \mathcal{W}_F \rightarrow \{ q \mathbb{Z} \} \text{ homomorphism} \]

\[ \| \cdot \| : \mathcal{I}_F = 1, \quad \| \mathcal{F}_F \| = q \]

Langlands parameter starts with

\[ \phi_I : \mathcal{I}(F) \rightarrow \mathcal{L}_G = \mathbb{G}_x \times \Gamma \]

\[ \phi(F_r) \text{ is semisimple} \]

\[ \phi_0 : \mathcal{W}_F \rightarrow \mathcal{L}_G \]

\[ \phi_0 \text{ generated by } \phi_I, \phi(F_r) \]
Details B: p-adic case

Langlands parameter starts with $\varphi_\pi: I(F) \to L_G = \gamma_G \times \Gamma$

$\gamma_G \subset$ reductive subgroup $G$

$\gamma_G = \text{fixed points of } \varphi(F)$

$\gamma_G$ is reductive; $G$ is semisimple

$\gamma_G$ is $\mathbb{Z}$-graded; $0$ subspace is $\gamma_G^0$.

(Waldarski) Langlands parameter $\varphi: W_E \to L_G$

is $\varphi_0: W_E \to L_G$ AND $N \in \gamma_G^{\pi, 0, 1}$, nilpotent element for $\gamma_G^{\pi, 1}$.
Details: real case

Wednesday, September 8, 2021 5:52 AM

\[ \Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\} \]

\[ L_G = V_G \times \Gamma = V_G \sqcup (V_G \sigma) \]

What matters about semisimple \( \lambda \in \mathfrak{g} \)
are integer eigenspaces of \( \text{ad}(\lambda) \)

\[ V_G e(\lambda) \]

pseudo levi in \( V_G \)

\[ \text{analogous to } \mathfrak{g}_\mathbb{C} \phi \mathbb{F}_\mathbb{C} = \langle \mathbb{Z} \rangle \text{-eigenspaces of } \phi(\mathbb{F}_\mathbb{C}) \text{ on } \mathfrak{g}_\mathbb{C} \phi \mathbb{F}_\mathbb{C} \]

\[ V_G \mathcal{L} \]

Levi subgp of \( V_G e(\lambda) \) and \( V_G \)

\[ \text{CANONICAL FLAT of } \lambda \text{ is } \Lambda = \lambda + \left[ \sum_j \text{1-eigenspace of } \text{ad}(\lambda) \right] C G \cdot \lambda \]

\[ e \cdot \text{constant } \]

\[ \text{on } \Delta \text{ } e(\lambda) = e(\lambda) \]

\[ \mathfrak{g}_2(\lambda) \text{ nilpotent} \]

\[ \text{LANGLANDS PARAMETER is } (j, \lambda), j \in LG \setminus V_G \quad \land \text{c } \mathfrak{g} \text{ canonical flat } j^2 = e(\lambda) \]
What happened to parabolic induction?

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\begin{align*}
\Gamma & \text{- fixed character } \xi \text{ of } H(F) \text{ torus in } G(F) \\
\downarrow \quad & \\
\Gamma & \text{- fixed one-parameter subgroup } C^* \rightarrow G \\
\nu \cdot L & = \text{Cent}_G(\xi(C^*))
\end{align*}
\]

\[
\text{rational parabolic} \\
P(F) = L(F) \cup \mathbb{H}(F) \\
H(F)
\]

rational