MACDONALD POLYNOMIALS AND DECOMPOSITION NUMBERS FOR FINITE UNITARY GROUPS

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September 2020

Finite general linear groups

$$GL_n(q) = \{ M \in \mathrm{Mat}_{n\times n}(\mathbb{F}_q) \mid \det M \neq 0 \}$$

Finite general unitary groups

$$GU_n(q) = \{ M \in \mathrm{Mat}_{n\times n}(\mathbb{F}_{q^2}) \mid M\, {}^t\overline{M} = \mathrm{Id} \}$$

where $\overline{a} = a^q$ for $a \in \mathbb{F}_{q^2}$

Unipotent representations

The irreducible unipotent representations of $GL_n(q)$ or $GU_n(q)$ are labelled by partitions of n

* over $\mathbb{C}$: unipotent characters $\{\Delta(\lambda)\}_{\lambda \vdash n}$

  e.g $\Delta(n)$ = trivial character
  $\Delta(1^n)$ = Steinberg character

* over $\overline{\mathbb{F}}_\ell$ with $\ell \nmid q$ $\{L(\lambda)\}_{\lambda \vdash n}$

Main Problem

$\Delta(\lambda)$ defined over $\mathbb{Q}$ $\supseteq$ lattice over $\mathbb{Z}$ $\rightsquigarrow$ $\overline{\Delta}(\lambda)$ reduction mod. ℓ

$$\boxed{[\overline{\Delta}(\lambda) : L(\mu)] = \,??}$$

"decomposition numbers"

Solved for $GL_n(q)$ for $\ell \gg 0$ (LLT algorithm)
Unknown for $GU_n(q)$! ($n \geqslant 11$)

ENNOLA DUALITY

$$\text{"} GU_n(q) = GL_n(-q) \text{"}$$

- Polynomial order of the groups

$$|GL_n(q)| = \prod_{i=0}^{n-1} (q^n - q^i) \qquad |GU_n(q)| = \pm \prod ((-q)^n - (-q)^i)$$

- Unipotent characters and their values

$$\dim(\Delta^{GU}(\lambda)) = \pm \dim(\Delta^{GL}(\lambda))|_{q \leftrightarrow -q}$$

<u>example</u>: $n = 4$

λ	$\dim \Delta^{GL}(\lambda)$	$\dim \Delta^{GU}(\lambda)$
(4)	1	1
(31)	$q(q^2+q+1)$	$q(q^2-q+1)$
(2^2)	$q^2(q^2+1)$	$q^2(q^2+1)$
(21^2)	$q^3(q^2+q+1)$	$q^3(q^2-q+1)$
(1^4)	q^6	q^6

extends to values on other unipotent classes.

- unipotent blocks over $\overline{\mathbb{F}}_\ell$ for $GL_n(q)$

  $\updownarrow$

  ________________ $\overline{\mathbb{F}}_{\ell'}$ for $GU_n(q)$

order of q in $\mathbb{F}_\ell^\times$ = order of $-q$ in $\mathbb{F}_{\ell'}^\times$

- decomposition numbers ???

$$\begin{array}{cc} & GL_4(q) \\ \begin{array}{c}(4)\\(31)\\(2^2)\\(21^2)\\(1^4)\end{array} & \begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{bmatrix} \\ & \ell \mid q-1 \end{array} \quad \overset{!!!}{\neq} \quad \begin{array}{c} GU_4(q) \\ \begin{bmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 1 & 1 & . & . \\ . & . & 1 & 1 & . \\ . & 1 & 1 & 3 & 1 \end{bmatrix} \\ \ell \mid q+1 \end{array}$$

Ennola duality ⇝ derived equivalence

$$\boxed{D^b(GL_n(-q)\text{-mod}) \overset{E}{\simeq} D^b(GU_n(q)\text{-mod})}$$

$$\Delta(\lambda) \longmapsto \Delta(\lambda)\,[\cdots]$$
$$L(\lambda) \longmapsto ???$$ COMPUTABLE ?

First problem: make sense of $GL_n(-q)$-mod

Second problem: construct E

The case $\ell \mid q-1$ $\quad (\ell' \mid q+1)$

$q-1 = \ell^r m$ with $m \wedge \ell = 1$ $\qquad \Bbbk = \mathbb{F}_\ell$

Principal block of $\Bbbk GL_n(q)$ $\overset{\text{Morita}}{\simeq}$ $\underbrace{\Bbbk(\mathbb{Z}/\ell^r\mathbb{Z})^n}_{\Bbbk[x_1,\ldots,x_r]/x_i^{\ell^r}} \rtimes S_n$

$\ell^r \to \infty$ ⇝ $\underbrace{(S(V)\otimes\Lambda(V))}_{\text{cohomology of } (\mathbb{Z}/\ell^r\mathbb{Z})^n \text{ over } \Bbbk} \rtimes S_n$ with $V = \Bbbk^n$

Representations of $(S(V)\otimes\Lambda(V))\rtimes S_n$ $\circlearrowright\ \nabla$ (perverse)

$\updownarrow$ Koszul

——————— $(S(V)\otimes S(V^*))\rtimes S_n$

$\updownarrow$

Coherent sheaves on $(V\oplus V^*)/S_n$ (here $V=\mathbb{C}^n$)

$\updownarrow$ $\qquad\qquad\qquad\uparrow$

Coherent sheaves on $\underbrace{\mathrm{Hilb}^n\,\mathbb{C}^2}_{\text{ideals of } \mathbb{C}[x,y] \text{ of length } n}$ $\circlearrowright\ -\otimes\mathcal{O}(1)$

$\Longrightarrow$ a bigraded self derived equivalence ∇ of the model for the principal block of $Gl_n(q)$

<u>Fact</u> : $\boxed{\nabla \text{ has the same behavior as } E^2}$

in Deligne-Lusztig theory

$$\begin{tikzcd} H^\bullet(X(w)) \arrow[r, "E"] \arrow[rr, bend right, "\nabla"'] & H^\bullet(X(w_0 w)) \arrow[r, "E"] & H^\bullet(X(w_0^2 w)) \end{tikzcd}$$

DL variety for $w\in S_n$

$\underline{\mathrm{Ex}}$: $n = 2$

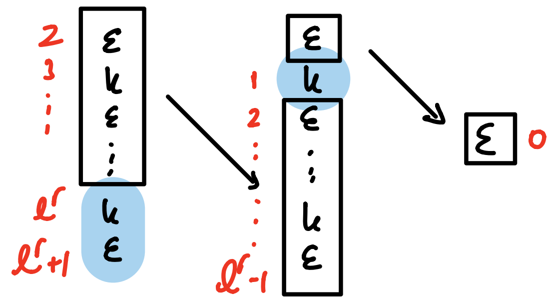
$$H = \left(\left(\mathbb{Z}/\ell^r\mathbb{Z} \right)^2 \rtimes \mathbb{Z}/2\mathbb{Z} \right) / \, \mathbb{Z}/\ell^r\mathbb{Z}$$

$$\simeq \mathbb{Z}/\ell^r\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$$

$\mathrm{Irr}\, H = \{ \Bbbk, \varepsilon \}$

E^2 perverse of perversity $\Bbbk \mapsto 2$, $\varepsilon \mapsto 0$

$E^2(\Bbbk) = \Bbbk[-2]$ $\qquad$ $E^2(\varepsilon) = \overset{0}{P_\varepsilon} \longrightarrow \overset{1}{P_\varepsilon} \twoheadrightarrow \overset{2}{\varepsilon}$



grading = radical series

$$[E^2] \longleftrightarrow \begin{bmatrix} u^{\ell^r+1} & \cdot \\ u^{\ell^r} - u & 1 \end{bmatrix}$$ in the basis $\mathrm{Irr}\, H$

$$[\nabla] \longleftrightarrow \begin{bmatrix} u & \cdot \\ 1-uv & v \end{bmatrix} = \begin{bmatrix} v & \cdot \\ \cdot & v \end{bmatrix} \begin{bmatrix} uv^{-1} & \cdot \\ v^{-1}-u & 1 \end{bmatrix}$$

$[E^2] \sim [\nabla]$ with $v = u^{-1-\ell^r}$

[Bergeron–Garsia–Haiman]

$\boxed{\nabla \text{ computable in the Grothendieck group}}$

$\mathbb{C}^\times \times \mathbb{C}^\times \curvearrowright \mathrm{Hilb}^n \mathbb{C}^2$

fixed points $\{I_\lambda\}_{\lambda \vdash n}$ parametrized by partitions

$- \otimes \mathcal{O}(1)$ acts diagonally on skyscraper sheaves supported on fixed points

[Haiman] The class of I_λ in the Grothendieck group is the Macdonald polynomial $\tilde{H}_\lambda$

$$\begin{tikzcd}
\bigoplus_{n \geq 0} \mathbb{Q} \otimes_{\mathbb{Z}} K_0(\mathrm{Coh}_{\mathbb{C}^\times \times \mathbb{C}^\times} \mathrm{Hilb}^n \mathbb{C}^2) \arrow[dd, "\wr"] & & \simeq \mathbb{Q}[u^{\pm 1}, v^{\pm 1}]\,\mathrm{Sym} \\
\ni \left[\begin{array}{c}\text{Skyscraper} \\ \text{sheaf at } I_\lambda\end{array}\right] & & \tilde{H}_\lambda \ \text{Macdonald} \arrow[ll] \arrow[d, dashed, leftrightarrow, "\text{known!}"] \ \ \ni \\
\ni [\Bbbk \otimes \Bbbk \otimes \chi_\lambda] & & s_\lambda\left[\frac{X}{1-u}\right] \ \text{Schur} \arrow[ll] \\
\bigoplus_{n \geq 0} \mathbb{Q} \otimes_{\mathbb{Z}} K_0((S(V) \otimes \Lambda(V)) \rtimes S_n\text{-mod}^{gr}) & &
\end{tikzcd}$$

$[\nabla]$ diagonal in the basis $\{\tilde{H}_\lambda\}_{\lambda \vdash n}$

$[\nabla]$ in the basis $\{s_\lambda[\frac{X}{1-u}]\}_{\lambda \vdash n}$ (irreps) computable

$\underline{n=3}$: on the basis of irreps

$$[\nabla] = \begin{bmatrix} u^3 & \cdot & \cdot \\ (u+u^2)(1-uv) & uv & \cdot \\ (1-uv)(1-uv(u+v)) & v(1+v)(1-uv) & v^3 \end{bmatrix}$$

$\underline{\text{Rmk}}$: $[\nabla]_{u,v=1} = \mathrm{Id}$

$\underline{\text{Question}}$: $\nabla \overset{??}{=} E^2 = E_{GL \to GU} E_{GU \to GL}$

$\rightsquigarrow$ formal construction of the principal ℓ-block of $GU_n(q)$ when $\ell \mid q+1$

$\rightsquigarrow$ unique factorisation of $[\nabla]$ as

$$[\nabla] = [E] \cdot [E]^{-1}_{u,v = u^{-1}, v^{-1}}$$

with some positivity properties for $[E]$

Example : $n=3$

$$[E] = \begin{bmatrix} u\sqrt{u} & \cdot & \cdot \\ \sqrt{u}(1-uv) & \sqrt{uv} & \cdot \\ u\sqrt{u}(uv-v-1) & -(1+v)v\sqrt{uv} & v\sqrt{v} \end{bmatrix}$$

$$[E]_{1,1} = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ -1 & -2 & 1 \end{bmatrix} \qquad \begin{array}{c}\text{dec. mat}\\ \text{of } GU_3(q)\end{array} = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ 1 & 2 & 1 \end{bmatrix}$$

CONJECTURE (D-Rouquier)

$[E]_{\sqrt{u},\sqrt{v}=1}$ is, up to signs, the decomposition matrix of the principal ℓ-block of $GU_n(q)$ when $\ell \mid q+1$ and $\ell > n$

More convincing example: $GU_9(9)$

$\rightsquigarrow$ 30×30 matrix for $[E]$

$$\text{Entry} = u\sqrt{uv}\,v^{12}\Big(-1-2v-4v^2+uv-6v^3+2uv^2-8v^4+4uv^3 -9v^5+5uv^4-9v^6+6uv^5-8v^7+5uv^6 -6v^8+4uv^7-4v^9+2uv^8-2v^{10}+uv^9-v^{11}\Big)$$

$$\xrightarrow{u,v=1} -30 = -\left[\Delta(1^9) : L(32^21^2)\right]$$