Spherical varieties, L-functions, and crystal bases

Jonathan Wang (joint w/ Yiannis Sakellaridis)

MIT

MIT Lie Groups Seminar, September 23, 2020

Notes available at:
http://jonathanpwang.com/notes/sphL_talk_notes.pdf

Outline

1. What is a spherical variety?
2. Function-theoretic results
3. Geometry $(k = \mathbb{C})$

- $F = \mathbb{F}_q((t))$, $O = \mathbb{F}_q[[t]]$
- $k = \overline{\mathbb{F}}_q$
- G connected (split) reductive group $/\mathbb{F}_q$

What is a spherical variety?

Definition

A G-variety $X_{/\mathbb{F}_q}$ is called spherical if X_k is normal and has an open dense orbit of $B_k \subset G_k$ after base change to k

What is a spherical variety?

Definition

A G-variety $X_{/\mathbb{F}_q}$ is called spherical if X_k is normal and has an open dense orbit of $B_k \subset G_k$ after base change to k

Think of this as a finiteness condition (good combinatorics)
($\Longleftrightarrow$ X has finitely many B-orbits)

What is a spherical variety?

Definition

A G-variety $X_{/\mathbb{F}_q}$ is called spherical if X_k is normal and has an open dense orbit of $B_k \subset G_k$ after base change to k

Think of this as a finiteness condition (good combinatorics)

Examples:

- Toric varieties $G = T$
- Symmetric spaces $K\backslash G$
  - Group $X = G' \circlearrowleft G' \times G' = G$

Why are they relevant?

Conjecture (Sakellaridis, Sakellaridis–Venkatesh)

For any affine spherical G-variety X (*),

and an irreducible unitary $G(F)$-representation π, there is an "integral"

$$|\mathcal{P}_X|^2_\pi : \pi \otimes \bar{\pi} \to \mathbb{C}$$

involving the IC function of $X(O)$ such that

1. $|\mathcal{P}_X|^2_\pi \neq 0$ *determines a functorial lifting of π to $\sigma \in \mathrm{Irr}(G_X(F))$ corresponding to a map $\check{G}_X(\mathbb{C}) \to \check{G}(\mathbb{C})$,*
2. *there should exist a $\check{G}_X$-representation*

$$\rho_X : \check{G}_X(\mathbb{C}) \to \mathrm{GL}(V_X)$$

such that $|\mathcal{P}_X|^2_\pi$ "$=$" $L(\sigma, \rho_X, s_0)$ for a special value s_0.

Why are they relevant?

Conjecture (Sakellaridis, Sakellaridis–Venkatesh)

For any affine spherical G-variety X (*),
and an irreducible unitary $G(F)$-representation π, there is an "integral"

$$|\mathcal{P}_X|^2_\pi : \pi \otimes \bar{\pi} \to \mathbb{C}$$

involving the IC function of $X(O)$ such that

1. *$|\mathcal{P}_X|^2_\pi \neq 0$ determines a functorial lifting of π to $\sigma \in \mathrm{Irr}(G_X(F))$ corresponding to a map $\check{G}_X(\mathbb{C}) \to \check{G}(\mathbb{C})$,*
2. *there should exist a $\check{G}_X$-representation*

$$\rho_X : \check{G}_X(\mathbb{C}) \to \mathrm{GL}(V_X)$$

such that $|\mathcal{P}_X|^2_\pi$ "$=$" $L(\sigma, \rho_X, s_0)$ for a special value s_0.

Why are they relevant?

Conjecture (Sakellaridis, Sakellaridis–Venkatesh)

For any affine spherical G-variety X (*), and an irreducible unitary $G(F)$-representation π, there is an "integral"

$$|\mathcal{P}_X|^2_\pi : \pi \otimes \bar{\pi} \to \mathbb{C}$$

involving the IC function of $X(O)$ such that

1. $|\mathcal{P}_X|^2_\pi \neq 0$ *determines a functorial lifting of* π *to* $\sigma \in \mathrm{Irr}(G_X(F))$ *corresponding to a map* $\check{G}_X(\mathbb{C}) \to \check{G}(\mathbb{C})$,
2. *there should exist a* $\check{G}_X$*-representation*

$$\rho_X : \check{G}_X(\mathbb{C}) \to \mathrm{GL}(V_X)$$

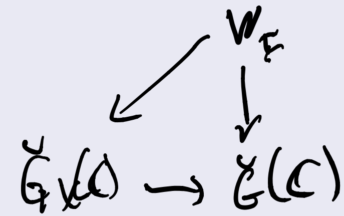
such that $|\mathcal{P}_X|^2_\pi$ "$=$" $L(\sigma, \rho_X, s_0)$ *for a special value* s_0.

Why are they relevant?

Conjecture (Sakellaridis, Sakellaridis–Venkatesh)

For any *affine* *spherical* *G-variety X (*),*
and an irreducible unitary $G(F)$-representation π, there is an "integral"

$$|\mathcal{P}_X|^2_\pi : \pi \otimes \bar{\pi} \to \mathbb{C}$$



involving the IC function of $X(O)$ such that

1. $|\mathcal{P}_X|^2_\pi \neq 0$ *determines a functorial lifting of π to $\sigma \in \mathrm{Irr}(G_X(F))$ corresponding to a map $\check{G}_X(\mathbb{C}) \to \check{G}(\mathbb{C})$,*
2. *there should exist a $\check{G}_X$-representation*

$$\rho_X : \check{G}_X(\mathbb{C}) \to \mathrm{GL}(V_X)$$

such that $|\mathcal{P}_X|^2_\pi$ "$=$" $L(\sigma, \rho_X, s_0)$ for a special value s_0.

Why are they relevant?

Conjecture (Sakellaridis, Sakellaridis–Venkatesh)

For any affine spherical G-variety X (*), and an irreducible unitary $G(F)$-representation π, there is an "integral"

$$|\mathcal{P}_X|^2_\pi : \pi \otimes \bar{\pi} \to \mathbb{C}$$

involving the IC function of $X(O)$ such that

1. $|\mathcal{P}_X|^2_\pi \neq 0$ *determines a functorial lifting of* π *to* $\sigma \in \mathrm{Irr}(G_X(F))$ *corresponding to a map* $\check{G}_X(\mathbb{C}) \to \check{G}(\mathbb{C})$, Later: $\check{G}_X = \check{G}$
2. *there should exist a* $\check{G}_X$*-representation*

$$\rho_X : \check{G}_X(\mathbb{C}) \to \mathrm{GL}(V_X)$$

such that $|\mathcal{P}_X|^2_\pi$ "=" $L(\sigma, \rho_X, s_0)$ *for a special value* s_0.

$\circlearrowleft G_X(F)$

Some history on $\check{G}_X$

Goal: a map $\check{G}_X \to \check{G}$ with finite kernel

Some history on $\check{G}_X$

Goal: a map $\check{G}_X \to \check{G}$ with finite kernel

- $\check{T}_X$ is easy to define

Some history on $\check{G}_X$

Goal: a map $\check{G}_X \to \check{G}$ with finite kernel

- $\check{T}_X$ is easy to define
- Little Weyl group W_X and spherical root system
    - Symmetric variety: Cartan '27

Some history on $\check{G}_X$

Goal: a map $\check{G}_X \to \check{G}$ with finite kernel

- $\check{T}_X$ is easy to define
- Little Weyl group W_X and spherical root system
  - Symmetric variety: Cartan '27
  - Spherical variety: Luna–Vust '83, Brion '90; reflection group of fundamental domain

Some history on $\check{G}_X$

Goal: a map $\check{G}_X \to \check{G}$ with finite kernel

- $\check{T}_X$ is easy to define
- Little Weyl group W_X and spherical root system
  - Symmetric variety: Cartan '27
  - Spherical variety: Luna–Vust '83, Brion '90; reflection group of fundamental domain
  - Irreducible G-variety: Knop '90, '93, '94; moment map, invariant differential operators

    $T^*X \longrightarrow \mathfrak{g}^*$ $\mathcal{D}(X)^G$
- Gaitsgory–Nadler '06: define subgroup $\check{G}_X^{GN} \subset \check{G}$ using Tannakian formalism

Some history on $\check{G}_X$

Goal: a map $\check{G}_X \to \check{G}$ with finite kernel

- $\check{T}_X$ is easy to define
- Little Weyl group W_X and spherical root system
  - Symmetric variety: Cartan '27
  - Spherical variety: Luna–Vust '83, Brion '90; reflection group of fundamental domain
  - Irreducible G-variety: Knop '90, '93, '94; moment map, invariant differential operators
- Gaitsgory–Nadler '06: define subgroup $\check{G}_X^{GN} \subset \check{G}$ using Tannakian formalism
- Sakellaridis–Venkatesh '12: normalized root system, define $\check{G}_X \to \check{G}$ combinatorially with image $\check{G}_X^{GN}$ under assumptions about GN

Some history on $\check{G}_X$

Goal: a map $\check{G}_X \to \check{G}$ with finite kernel

- $\check{T}_X$ is easy to define
- Little Weyl group W_X and spherical root system
  - Symmetric variety: Cartan '27
  - Spherical variety: Luna–Vust '83, Brion '90; reflection group of fundamental domain
  - Irreducible G-variety: Knop '90, '93, '94; moment map, invariant differential operators
- Gaitsgory–Nadler '06: define subgroup $\check{G}_X^{GN} \subset \check{G}$ using Tannakian formalism
- Sakellaridis–Venkatesh '12: normalized root system, define $\check{G}_X \to \check{G}$ combinatorially with image $\check{G}_X^{GN}$ under assumptions about GN
- Knop–Schalke '17: define $\check{G}_X \to \check{G}$ combinatorially unconditionally

	$X \circlearrowleft G$	$\check{G}_X$	V_X
Usual Langlands	$G' \circlearrowleft G' \times G'$	$\check{G}'$	$\check{\mathfrak{g}}'$

	$X \circlearrowleft G$	$\check{G}_X$	V_X
Usual Langlands	$G' \circlearrowleft G' \times G'$	$\check{G}'$	$\check{\mathfrak{g}}'$
Whittaker normalization	$(N, \psi)\backslash G$	$\check{G}$	0

$$C^\infty((N,\psi)\backslash G) = C^\infty(G)^{N,\psi}$$

$T^* U = U \oplus U^*$

	$X \circlearrowleft G$	$\check{G}_X$	V_X
Usual Langlands	$G' \circlearrowleft G' \times G'$	$\check{G}'$	$\check{\mathfrak{g}}'$
Whittaker normalization	$(N, \psi)\backslash G$	$\check{G}$	0
Hecke	$\mathbb{G}_m\backslash \mathrm{PGL}_2$	$\check{G} = \mathrm{SL}_2$	$T^*\mathrm{std}$

	$X \circlearrowleft G$	$\check{G}_X$	V_X
Usual Langlands	$G' \circlearrowleft G' \times G'$	$\check{G}'$	$\check{\mathfrak{g}}'$
Whittaker normalization	$(N, \psi)\backslash G$	$\check{G}$	0
Hecke	$\mathbb{G}_m\backslash \mathrm{PGL}_2$	$\check{G} = \mathrm{SL}_2$	$T^*\mathrm{std}$
Rankin–Selberg, Jacquet–Piatetski-Shapiro–Shalika	$\overline{H\backslash \mathrm{GL}_n \times \mathrm{GL}_n} = \mathrm{GL}_n \times \mathbb{A}^n = X \circlearrowleft$	$\check{G}$	$T^*(\mathrm{std} \otimes \mathrm{std})$

$H =$ diagonal mirabolic $\quad \mathrm{GL}_n \times \mathrm{GL}_n = G$

	$X \circlearrowleft G$	$\check{G}_X$	V_X
Usual Langlands	$G' \circlearrowleft G' \times G'$	$\check{G}'$	$\check{\mathfrak{g}}'$
Whittaker normalization	$(N, \psi)\backslash G$	$\check{G}$	0
Hecke	$\mathbb{G}_m\backslash \mathrm{PGL}_2$	$\check{G} = \mathrm{SL}_2$	$T^*\mathrm{std}$
Rankin–Selberg, Jacquet–Piatetski-Shapiro–Shalika	$\overline{H\backslash \mathrm{GL}_n \times \mathrm{GL}_n} = \mathrm{GL}_n \times \mathbb{A}^n$	$\check{G}$	$T^*(\mathrm{std} \otimes \mathrm{std})$
Gan–Gross–Prasad	$\mathrm{SO}_{2n}\backslash \mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n}$	$\check{G} = \mathrm{SO}_{2n} \times \mathrm{Sp}_{2n}$	$\mathrm{std} \otimes \mathrm{std}$

Example (Sakellaridis)

$G = \mathrm{GL}_2^{\times n} \times \mathbb{G}_m$, $H =$

$$\left\{ \begin{pmatrix} a & x_1 \\ & 1 \end{pmatrix} \times \begin{pmatrix} a & x_2 \\ & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} a & x_n \\ & 1 \end{pmatrix} \times a \,\middle|\, x_1 + \cdots + x_n = 0 \right\}$$

$X = \overline{H \backslash G}$

Example (Sakellaridis)

$G = \mathrm{GL}_2^{\times n} \times \mathbb{G}_m$, $H =$

$$\left\{ \begin{pmatrix} a & x_1 \\ & 1 \end{pmatrix} \times \begin{pmatrix} a & x_2 \\ & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} a & x_n \\ & 1 \end{pmatrix} \times a \,\middle|\, x_1 + \cdots + x_n = 0 \right\}$$

$X = \overline{H \backslash G}$

- $\check{G}_X = \check{G} = \mathrm{GL}_2^{\times n} \times \mathbb{G}_m$
- $V_X = T^*(\mathrm{std}_2^{\otimes n} \otimes \mathrm{std}_1)$.

Example (Sakellaridis)

$G = \mathrm{GL}_2^{\times n} \times \mathbb{G}_m$, $H =$

$$\left\{ \begin{pmatrix} a & x_1 \\ & 1 \end{pmatrix} \times \begin{pmatrix} a & x_2 \\ & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} a & x_n \\ & 1 \end{pmatrix} \times a \,\middle|\, x_1 + \cdots + x_n = 0 \right\}$$

$X = \overline{H \backslash G}$

- $\check{G}_X = \check{G} = \mathrm{GL}_2^{\times n} \times \mathbb{G}_m$
- $V_X = T^*(\mathrm{std}_2^{\otimes n} \otimes \mathrm{std}_1)$.

To find new interesting examples, need to consider singular $X \neq H \backslash G$.

Theorem (Luna, Richardson)

$H \backslash G$ is affine if and only if H is reductive

$\check{G}_X = \check{G}$

Avoid: $O_n \backslash GL_n$

For this talk, assume $\check{G}_X = \check{G}$ (and X has no type N roots). ['N' is for normalizer]

$\check{G}_X = \check{G}$

For this talk, assume $\check{G}_X = \check{G}$ (and X has no type N roots). ['N' is for normalizer]

Equivalent to:

(Base change to k)

- X has open B-orbit $X^\circ \cong B$ $\qquad x_0 \in X^\circ(\mathbb{F}_q)$
- $X^\circ P_\alpha/\mathcal{R}(P_\alpha) \cong \mathbb{G}_m\backslash \mathrm{PGL}_2$ for every simple α, $P_\alpha \supset B$

$$P_\alpha / \mathcal{R}(P_\alpha) = PGL_2$$

Sakellaridis–Venkatesh á la Bernstein

asymptotics

Definition

Fix $x_0 \in X^\circ(\mathbb{F}_q)$ in open B-orbit. Define the X-Radon transform

$$\pi_! : C_c^\infty(X(F))^{G(O)} \hookrightarrow C^\infty(N(F)\backslash G(F))^{G(O)}$$

Sakellaridis–Venkatesh á la Bernstein

Definition

Fix $x_0 \in X^\circ(\mathbb{F}_q)$ in open B-orbit. Define the X-Radon transform

$$\pi_! : C_c^\infty(X(F))^{G(O)} \to C^\infty(N(F)\backslash G(F))^{G(O)}$$

by

$$\pi_!\Phi(g) := \int_{N(F)} \Phi(x_0 n g) dn, \quad g \in G(F)$$

Sakellaridis–Venkatesh á la Bernstein

Definition

Fix $x_0 \in X^\circ(\mathbb{F}_q)$ in open B-orbit. Define the X-Radon transform

$$\pi_! : C_c^\infty(X(F))^{G(O)} \to C^\infty(N(F)\backslash G(F))^{G(O)}$$

by

$$\pi_!\Phi(g) := \int_{N(F)} \Phi(x_0 n g) d n, \quad g \in G(F)$$

$\pi_!\Phi$ is a function on $N(F)\backslash G(F)/G(O) = T(F)/T(O) = \check{\Lambda}$.

Sakellaridis–Venkatesh á la Bernstein

Definition

Fix $x_0 \in X^\circ(\mathbb{F}_q)$ in open B-orbit. Define the X-Radon transform

$$\pi_! : C_c^\infty(X(F))^{G(O)} \to C^\infty(N(F)\backslash G(F))^{G(O)}$$

by

$$\pi_!\Phi(g) := \int_{N(F)} \Phi(x_0 n g) dn, \quad g \in G(F)$$

$\pi_!\Phi$ is a function on $N(F)\backslash G(F)/G(O) = T(F)/T(O) = \check{\Lambda}$.

Related:

- spherical functions (unramified Hecke eigenfunction) on $X(F)$
- unramified Plancherel measure on $X(F)$

Conjecture 1 (Sakellaridis–Venkatesh)

Assume $\check{G}_X = \check{G}$ and X has no type N roots.

Conjecture 1 (Sakellaridis–Venkatesh)

Assume $\check{G}_X = \check{G}$ *and* X *has no type* N *roots.*
There exists a symplectic $V_X \in \mathrm{Rep}(\check{G})$ *with a* $\check{T}$ *polarization* $V_X = V_X^+ \oplus (V_X^+)^*$ *such that*

Conjecture 1 (Sakellaridis–Venkatesh)

Assume $\check{G}_X = \check{G}$ and X has no type N roots.
There exists a symplectic $V_X \in \mathrm{Rep}(\check{G})$ with a $\check{T}$ polarization $V_X = V_X^+ \oplus (V_X^+)^*$ such that

$$\pi_! \Phi_{\mathrm{IC}_{X(O)}} = \frac{\prod_{\check{\alpha} \in \check{\Phi}_G^+} (1 - q^{-1} e^{\check{\alpha}})}{\prod_{\check{\lambda} \in \mathrm{wt}(V_X^+)} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})} \in \mathrm{Fn}(\check{\Lambda})$$

where $e^{\check{\lambda}}$ is the indicator function of $\check{\lambda}$, $e^{\check{\lambda}} e^{\check{\mu}} = e^{\check{\lambda} + \check{\mu}}$

$$\frac{1}{1 - q^{-\frac{1}{2}} e^{\check{\lambda}}} = \sum_{n \geq 0} (q^{-\frac{1}{2}} e^{\check{\lambda}})^n$$

Conjecture 1 (Sakellaridis–Venkatesh)

Assume $\check{G}_X = \check{G}$ *and* X *has no type N roots.* $\check{G} \times V_X \rightsquigarrow \check{\mathfrak{g}}^*$

There exists a symplectic $V_X \in \mathrm{Rep}(\check{G})$ *with a* $\check{T}$ *polarization* $V_X = V_X^+ \oplus (V_X^+)^*$ *such that*

$$\pi_! \Phi_{\mathrm{IC}_{X(O)}} = \frac{\prod_{\check{\alpha} \in \check{\Phi}_G^+} (1 - q^{-1} e^{\check{\alpha}})}{\prod_{\check{\lambda} \in \mathrm{wt}(V_X^+)} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})} \in \mathrm{Fn}(\check{\Lambda})$$

$T(F)/T(O)$

where $e^{\check{\lambda}}$ *is the indicator function of* $\check{\lambda}$, $e^{\check{\lambda}} e^{\check{\mu}} = e^{\check{\lambda} + \check{\mu}}$

Mellin transform of right hand side gives

$e^{\check{\lambda}} \rightsquigarrow \check{\lambda}(\chi)$

$$\chi \in \check{T}(\mathbb{C}) \mapsto \frac{L(\chi, V_X^+, \frac{1}{2})}{L(\chi, \check{\mathfrak{n}}, 1)}, \text{ this is "half" of } \frac{L(\chi, V_X, \frac{1}{2})}{L(\chi, \check{\mathfrak{g}}/\check{\mathfrak{t}}, 1)}$$

← normalization

Warning: V_X^+ never $\check{G}$-rep

e.g. Hecke $V_X = T^* \mathrm{std}$, V_X^+ (1,0) (0,−1)

Previous work

Conjecture 1 (possibly with $\check{G}_X \neq \check{G}$) was proved in the following cases:

Previous work

Conjecture 1 (possibly with $\check{G}_X \neq \check{G}$) was proved in the following cases:

- Sakellaridis ('08, '13): function-theoretic ↙ smooth
  - $X = H\backslash G$ and H is reductive (iff $H\backslash G$ is affine), no assumption on $\check{G}_X$
  - doesn't consider $X \supsetneq H\backslash G$

Previous work

Conjecture 1 (possibly with $\check{G}_X \neq \check{G}$) was proved in the following cases:

- Sakellaridis ('08, '13):
  - $X = H\backslash G$ and H is reductive (iff $H\backslash G$ is affine), no assumption on $\check{G}_X$
  - doesn't consider $X \supsetneq H\backslash G$
- Braverman–Finkelberg–Gaitsgory–Mirković [BFGM] '02:
  - $X = \overline{N^-\backslash G}$, $\check{G}_X = \check{T}$, $V_X = \check{\mathfrak{n}}$

Previous work

Conjecture 1 (possibly with $\check{G}_X \neq \check{G}$) was proved in the following cases:

- Sakellaridis ('08, '13):
  - $X = H\backslash G$ and H is reductive (iff $H\backslash G$ is affine), no assumption on $\check{G}_X$
  - doesn't consider $X \supsetneq H\backslash G$
- Braverman–Finkelberg–Gaitsgory–Mirković [BFGM] '02:
  - $X = \overline{N^-\backslash G}$, $\check{G}_X = \check{T}$, $V_X = \check{\mathfrak{n}}$
- Bouthier–Ngô–Sakellaridis [BNS] '16:
  - X toric variety, $G = T$, $\check{G}_X = \check{T}$, weights of V_X correspond to lattice generators of a cone

Previous work

Conjecture 1 (possibly with $\check{G}_X \neq \check{G}$) was proved in the following cases:

- Sakellaridis ('08, '13): function-theoretic
  - $X = H\backslash G$ and H is reductive (iff $H\backslash G$ is affine), no assumption on $\check{G}_X$
  - doesn't consider $X \supsetneq H\backslash G$
- Braverman–Finkelberg–Gaitsgory–Mirković [BFGM] '02:
  - $X = \overline{N^- \backslash G}$, $\check{G}_X = \check{T}$, $V_X = \check{\mathfrak{n}}$
- Bouthier–Ngô–Sakellaridis [BNS] '16: geometric
  - X toric variety, $G = T$, $\check{G}_X = \check{T}$, weights of V_X correspond to lattice generators of a cone
  - $X \supset G'$ is L-monoid, $G = G' \times G'$, $\check{G}_X = \check{G}'$, $V_X = \check{\mathfrak{g}}' \oplus V^{\check{\lambda}}$

Theorem (Sakellaridis–W)

Assume X affine spherical, $\check{G}_X = \check{G}$ and X has no type N roots. Then

Theorem (Sakellaridis–W)

Assume X affine spherical, $\check{G}_X = \check{G}$ and X has no type N roots. Then

$$\pi_! \Phi_{\mathrm{IC}_{X(O)}} = \frac{\prod_{\check{\alpha} \in \check{\Phi}_G^+} (1 - q^{-1} e^{\check{\alpha}})}{\prod_{\check{\lambda} \in \mathrm{wt}(V_X^+)} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})}$$

for some $V_X^+ \in \mathrm{Rep}(\check{T})$ such that:

Theorem (Sakellaridis–W)

Assume X affine spherical, $\check{G}_X = \check{G}$ and X has no type N roots. Then

$$\pi_! \Phi_{\mathrm{IC}_{X(O)}} = \frac{\prod_{\check{\alpha} \in \check{\Phi}_G^+} (1 - q^{-1} e^{\check{\alpha}})}{\prod_{\check{\lambda} \in \mathrm{wt}(V_X^+)} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})}$$

for some $V_X^+ \in \mathrm{Rep}(\check{T})$ *such that:*

1. $V_X' := V_X^+ \oplus (V_X^+)^*$ *has action of* $(\mathrm{SL}_2)_\alpha$ *for every simple root* α
   - *We do not check Serre relations*

Theorem (Sakellaridis–W)

Assume X affine spherical, $\check{G}_X = \check{G}$ and X has no type N roots. Then

$$\pi_! \Phi_{\mathrm{IC}_{X(O)}} = \frac{\prod_{\check{\alpha} \in \check{\Phi}_G^+} (1 - q^{-1} e^{\check{\alpha}})}{\prod_{\check{\lambda} \in \mathrm{wt}(V_X^+)} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})}$$

for some $V_X^+ \in \mathrm{Rep}(\check{T})$ such that:

1. $V_X' := V_X^+ \oplus (V_X^+)^*$ *has action of* $(\mathrm{SL}_2)_\alpha$ *for every simple root* α
   - *We do not check Serre relations*
2. *Assuming V_X' satisfies Serre relations (so it is a $\check{G}$-rep), we determine its highest weights with multiplicities (in terms of X)*

Theorem (Sakellaridis–W)

Assume X affine spherical, $\check{G}_X = \check{G}$ and X has no type N roots. Then

$$\pi_! \Phi_{\mathrm{IC}_{X(O)}} = \frac{\prod_{\check{\alpha} \in \check{\Phi}_G^+} (1 - q^{-1} e^{\check{\alpha}})}{\prod_{\check{\lambda} \in \mathrm{wt}(V_X^+)} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})}$$

for some $V_X^+ \in \mathrm{Rep}(\check{T})$ such that:

1. $V_X' := V_X^+ \oplus (V_X^+)^*$ *has action of* $(\mathrm{SL}_2)_\alpha$ *for every simple root* α
   - *We do not check Serre relations*
2. *Assuming V_X' satisfies Serre relations (so it is a $\check{G}$-rep), we determine its highest weights with multiplicities (in terms of X)*

- (2) gives recipe for conjectural V_X in terms of X

Theorem (Sakellaridis–W)

Assume X affine spherical, $\check{G}_X = \check{G}$ and X has no type N roots. Then

$$\pi_! \Phi_{\mathrm{IC}_{X(O)}} = \frac{\prod_{\check{\alpha} \in \check{\Phi}_G^+} (1 - q^{-1} e^{\check{\alpha}})}{\prod_{\check{\lambda} \in \mathrm{wt}(V_X^+)} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})}$$

for some $V_X^+ \in \mathrm{Rep}(\check{T})$ such that:

1. $V_X' := V_X^+ \oplus (V_X^+)^*$ *has action of* $(\mathrm{SL}_2)_\alpha$ *for every simple root* α
   - *We do not check Serre relations* → reduce ~10 cases, G ss rank 2
2. *Assuming V_X' satisfies Serre relations (so it is a $\check{G}$-rep), we determine its highest weights with multiplicities (in terms of X)*

- (2) gives recipe for conjectural V_X in terms of X
- If V_X is minuscule, then $V_X = V_X'$.

Proposition

If $X = H \backslash G$ with H reductive, then V_X is minuscule.

Enter geometry

- Base change to $k = \overline{\mathbb{F}}_q$ (or $k = \mathbb{C}$)
- $\mathbf{X_O}(k) = X(k[\![t]\!])$
- $\mathbf{X_F}(k) = X(k(\!(t)\!))$

Enter geometry

- Base change to $k = \overline{\mathbb{F}}_q$ (or $k = \mathbb{C}$)
- $\mathbf{X_O}(k) = X(k[\![t]\!])$
- $\mathbf{X_F}(k) = X(k(\!(t)\!))$
- Problem: $\mathbf{X_O}$ is an infinite type scheme

no perverse sheaves

Enter geometry

- Base change to $k = \overline{\mathbb{F}}_q$ (or $k = \mathbb{C}$)
- $\mathbf{X_O}(k) = X(k[\![t]\!])$
- $\mathbf{X_F}(k) = X(k(\!(t)\!))$
- Problem: $\mathbf{X_O}$ is an infinite type scheme
- Bouthier–Ngô–Sakellaridis: IC function still makes sense by Grinberg–Kazhdan theorem

(draw O)

Zastava space

Drinfeld's proof of Grinberg–Kazhdan theorem gives an explicit model for $\mathbf{X_O}$:

$\mathcal{X}^0 \cong \mathcal{B}$

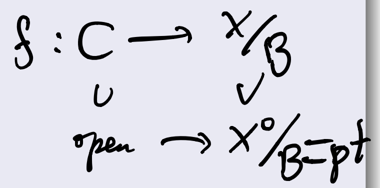
Zastava space

Drinfeld's proof of Grinberg–Kazhdan theorem gives an explicit model for $\mathbf{X_O}$:

Definition

Let C be a smooth curve over k. Define

$$\mathcal{Y} = \mathrm{Maps}_{\mathrm{gen}}(C, X/B \supset X^\circ/B)$$



Zastava space

Drinfeld's proof of Grinberg–Kazhdan theorem gives an explicit model for $\mathbf{X_O}$:

Definition

Let C be a smooth curve over k. Define

$$\mathcal{Y} = \mathrm{Maps}_{\mathrm{gen}}(C, X/B \supset X^\circ/B)$$

Following Finkelberg–Mirković, we call this the **Zastava space** of X.

$X = \overline{N^- \backslash G}$

Zastava space

Drinfeld's proof of Grinberg–Kazhdan theorem gives an explicit model for $\mathbf{X_O}$:

Definition

Let C be a smooth curve over k. Define

$$\mathcal{Y} = \mathrm{Maps}_{\mathrm{gen}}(C, X/B \supset X^\circ/B)$$

Following Finkelberg–Mirković, we call this the **Zastava space** of X.

Fact: $\mathcal{Y}$ is an infinite disjoint union of finite type schemes.

Zastava space

Drinfeld's proof of Grinberg–Kazhdan theorem gives an explicit model for $\mathbf{X_O}$:

Definition

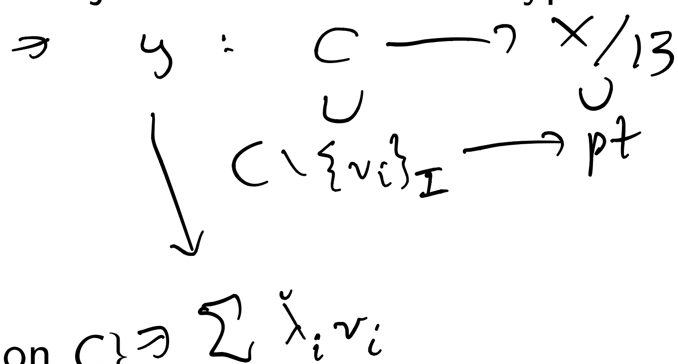
Let C be a smooth curve over k. Define

$$\mathcal{Y} = \mathrm{Maps}_{\mathrm{gen}}(C, X/B \supset X^\circ/B)$$

Following Finkelberg–Mirković, we call this the **Zastava space** of X.

Fact: $\mathcal{Y}$ is an infinite disjoint union of finite type schemes.

$$\mathcal{Y} \xrightarrow{\pi} \mathcal{A} \subset \{\check{\Lambda}\text{-valued divisors on } C\}$$



Define the **central fiber** $\mathbb{Y}^{\check\lambda} = \pi^{-1}(\check\lambda \cdot v)$ for a single point $v \in C(k)$.

$$\begin{array}{ccc} \mathcal{Y} & \longleftarrow & \mathbb{Y}^{\check\lambda} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ \mathcal{A} & \longleftarrow & \check\lambda \cdot v \end{array}$$

Define the **central fiber** $\mathbb{Y}^{\check{\lambda}} = \pi^{-1}(\check{\lambda} \cdot v)$ for a single point $v \in C(k)$.

$$\begin{array}{ccc} \mathcal{Y} & \longleftarrow & \mathbb{Y}^{\check{\lambda}} \\ \downarrow & & \downarrow \\ \mathcal{A} & \longleftarrow & \check{\lambda} \cdot v \end{array}$$

Graded factorization property

The fiber $\pi^{-1}(\check{\lambda}_1 v_1 + \check{\lambda}_2 v_2)$ for distinct v_1, v_2 is isomorphic to $\mathbb{Y}^{\check{\lambda}_1} \times \mathbb{Y}^{\check{\lambda}_2}$.

Define the **central fiber** $\mathbb{Y}^{\check{\lambda}} = \pi^{-1}(\check{\lambda} \cdot v)$ for a single point $v \in C(k)$.

$$\begin{array}{ccc} \mathcal{Y} & \longleftarrow & \mathbb{Y}^{\check{\lambda}} \\ \downarrow & & \downarrow \\ \mathcal{A} & \longleftarrow & \check{\lambda} \cdot v \end{array}$$

Graded factorization property

The fiber $\pi^{-1}(\check{\lambda}_1 v_1 + \check{\lambda}_2 v_2)$ for distinct v_1, v_2 is isomorphic to $\mathbb{Y}^{\check{\lambda}_1} \times \mathbb{Y}^{\check{\lambda}_2}$.

Upshot

$$\pi_! \Phi_{\mathrm{IC}_{\mathbf{x_0}}}(t^{\check{\lambda}}) = tr(\mathrm{Fr}, (\pi_! \mathrm{IC}_{\mathcal{Y}})|^*_{\check{\lambda} \cdot v})$$

Semi-small map

Can compactify π to proper map $\bar{\pi} : \overline{\mathcal{Y}} \to \mathcal{A}$. special $\check{G}_X = \check{G}$

Theorem (Sakellaridis–W)

Under previous assumptions, $\bar{\pi} : \overline{\mathcal{Y}} \to \mathcal{A}$ is stratified semi-small. In particular, $\bar{\pi}_! \mathrm{IC}_{\overline{\mathcal{Y}}}$ is perverse.

Semi-small map

Can compactify π to proper map $\bar{\pi} : \overline{\mathcal{Y}} \to \mathcal{A}$.

Theorem (Sakellaridis–W)

Under previous assumptions, $\bar{\pi} : \overline{\mathcal{Y}} \to \mathcal{A}$ is stratified semi-small. In particular, $\bar{\pi}_! \mathrm{IC}_{\overline{\mathcal{Y}}}$ is perverse.

If $\overline{\mathcal{Y}}$ is smooth, then semi-smallness amounts to the inequality

$$\dim \overline{\mathbb{Y}}^{\check{\lambda}} \leq \mathrm{crit}(\check{\lambda})$$

Semi-small map

Can compactify π to proper map $\bar{\pi} : \overline{\mathcal{Y}} \to \mathcal{A}$.

Theorem (Sakellaridis–W)

Under previous assumptions, $\bar{\pi} : \overline{\mathcal{Y}} \to \mathcal{A}$ is stratified semi-small. In particular, $\bar{\pi}_! \mathrm{IC}_{\overline{\mathcal{Y}}}$ is perverse.

If $\overline{\mathcal{Y}}$ is smooth, then semi-smallness amounts to the inequality

$$\dim \overline{\mathbb{Y}}^{\check{\lambda}} \leq \mathrm{crit}(\check{\lambda})$$

Decomposition theorem + factorization property imply

Euler product

$$tr(\mathrm{Fr}, (\bar{\pi}_! \mathrm{IC}_{\overline{\mathcal{Y}}})|^*_{?\cdot v}) = \frac{1}{\prod_{\check{\lambda} \in \mathfrak{B}^+} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})}$$

$\overline{\mathcal{Y}}$ vs $\mathcal{Y}$

Semi-small map

Can compactify π to proper map $\bar{\pi} : \overline{\mathcal{Y}} \to \mathcal{A}$.

Theorem (Sakellaridis–W)

Under previous assumptions, $\bar{\pi} : \overline{\mathcal{Y}} \to \mathcal{A}$ is stratified semi-small. In particular, $\bar{\pi}_! \mathrm{IC}_{\overline{\mathcal{Y}}}$ is perverse.

If $\overline{\mathcal{Y}}$ is smooth, then semi-smallness amounts to the inequality

$$\dim \overline{\mathbb{Y}}^{\check{\lambda}} \leq \mathrm{crit}(\check{\lambda})$$

Decomposition theorem + factorization property imply

Euler product

$$tr(\mathrm{Fr}, (\bar{\pi}_! \mathrm{IC}_{\overline{\mathcal{Y}}})|^*_{?\cdot v}) = \frac{1}{\prod_{\check{\lambda} \in \mathfrak{B}^+} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})}$$

relevant stratum supported at v

$\mathfrak{B}^+$ = irred. components of $\overline{\mathbb{Y}}^{\check{\lambda}}$ of dim = $\mathrm{crit}(\check{\lambda})$ as $\check{\lambda}$ varies

- $\mathfrak{B}^+ =$ irred. components of $\overline{\mathbb{Y}}^{\check{\lambda}}$ of dim $=$ crit$(\check{\lambda})$ as $\check{\lambda}$ varies

- $\mathfrak{B}^+ =$ irred. components of $\overline{\mathbb{Y}}^{\check{\lambda}}$ of dim $= \mathrm{crit}(\check{\lambda})$ as $\check{\lambda}$ varies
- Define V_X^+ to have basis $\mathfrak{B}^+$

- $\mathfrak{B}^+ =$ irred. components of $\overline{\mathbb{Y}}^{\check{\lambda}}$ of dim $= \mathrm{crit}(\check{\lambda})$ as $\check{\lambda}$ varies
- Define V_X^+ to have basis $\mathfrak{B}^+$
- Formally set $\mathfrak{B} = \mathfrak{B}^+ \sqcup (\mathfrak{B}^+)^*$, so $(\mathfrak{B}^+)^*$ is a basis of $(V_X^+)^*$

$\check{\lambda}$ $-\check{\lambda}$

- $\mathfrak{B}^+ =$ irred. components of $\overline{\mathbb{Y}}^{\check{\lambda}}$ of dim $= \mathrm{crit}(\check{\lambda})$ as $\check{\lambda}$ varies
- Define V_X^+ to have basis $\mathfrak{B}^+$
- Formally set $\mathfrak{B} = \mathfrak{B}^+ \sqcup (\mathfrak{B}^+)^*$, so $(\mathfrak{B}^+)^*$ is a basis of $(V_X^+)^*$

Theorem (Sakellaridis–W)

$\mathfrak{B}$ has the structure of a (Kashiwara) crystal, i.e., graph with weighted vertices and edges $\leftrightarrow$ raising/lowering operators $\tilde{e}_\alpha, \tilde{f}_\alpha$

- $\mathfrak{B}^+ =$ irred. components of $\overline{\mathbb{Y}}^{\check{\lambda}}$ of dim $= \mathrm{crit}(\check{\lambda})$ as $\check{\lambda}$ varies
- Define V_X^+ to have basis $\mathfrak{B}^+$
- Formally set $\mathfrak{B} = \mathfrak{B}^+ \sqcup (\mathfrak{B}^+)^*$, so $(\mathfrak{B}^+)^*$ is a basis of $(V_X^+)^*$

Theorem (Sakellaridis–W)

$\mathfrak{B}$ has the structure of a (Kashiwara) crystal, i.e., graph with weighted vertices and edges $\leftrightarrow$ raising/lowering operators $\tilde{e}_\alpha, \tilde{f}_\alpha$

Crystal basis is the (Lusztig) canonical basis at $q = 0$ of a f.d. $U_q(\check{\mathfrak{g}})$-module.

- $\mathfrak{B}^+ =$ irred. components of $\overline{\mathbb{Y}}^{\check{\lambda}}$ of dim $= \mathrm{crit}(\check{\lambda})$ as $\check{\lambda}$ varies
- Define V_X^+ to have basis $\mathfrak{B}^+$
- Formally set $\mathfrak{B} = \mathfrak{B}^+ \sqcup (\mathfrak{B}^+)^*$, so $(\mathfrak{B}^+)^*$ is a basis of $(V_X^+)^*$

Theorem (Sakellaridis–W)

$\mathfrak{B}$ has the structure of a (Kashiwara) crystal, i.e., graph with weighted vertices and edges $\leftrightarrow$ raising/lowering operators $\tilde{e}_\alpha, \tilde{f}_\alpha$

Crystal basis is the (Lusztig) canonical basis at $q = 0$ of a f.d. $U_q(\check{\mathfrak{g}})$-module.

$q=1$ $\qquad\qquad$ $q=0$

$$\text{f.d. } \check{G}\text{-representation} \rightsquigarrow \text{crystal basis} \in \{\text{crystals}\}$$

Conjecture 2

$\mathfrak{B}$ is the crystal basis for a finite dimensional $\check{G}$-representation V_X.

- Conjecture 2 implies Conjecture 1 ($\mathfrak{B} \leftrightarrow V'_X$).

Conjecture 2

$\mathfrak{B}$ is the crystal basis for a finite dimensional $\check{G}$-representation V_X.

- Conjecture 2 implies Conjecture 1 ($\mathfrak{B} \leftrightarrow V'_X$).
- Conjecture 2 resembles geometric constructions of crystal bases by Lusztig, Braverman–Gaitsgory, Kamnitzer involving irreducible components of Gr_G
- $\mathbb{Y}^{\check{\lambda}}, \overline{\mathbb{Y}}^{\check{\lambda}} \subset \mathrm{Gr}_G$

$S^\lambda = N_F t^\lambda \subset \mathrm{Gr}_G$

$X \supset \overline{P\backslash G}$

$\mathbb{Y}^\lambda \subset S^\lambda$

$\overline{\mathbb{Y}}^\lambda \subset \overline{S}^\lambda = \bigcup_{\mu \le \lambda} S^\mu$

$\overline{\mathbb{Y}}^\lambda \rightsquigarrow \overline{\mathbb{Y}}^\lambda \cap S^{\lambda-\alpha} \subset \mathbb{Y}^{\lambda-\alpha}$

$V_X = V_{\overline{H\backslash G}} \oplus \bigoplus \left(V^{\check\theta} \oplus (V^{\check\theta})^*\right)$

$\check\theta \in \check\Lambda_G$ ← determined by $X \supset \overline{H\backslash G}$