A strong Henniart identity for reductive groups over finite fields.

Charlotte Chan (MIT)

joint with Masao Oi (Kyoto)

Supercuspidal representations of p-adic GL_n

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Q: Can we compare these parametrizations?

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Thm. (Henniart) Let $\theta : E^\times \to \mathbb{C}^\times$ smooth, triv Gal(E/F)-stab.

Then $\pi'(\theta) \cong \pi(\theta\omega)$, where ω = unram char of $E^\times$ s.t.
$\omega(\varpi) = (-1)^{n-1}$

(Note: $\omega = \mathbb{1}$ if n odd, ω = unram quad. char. if n even)

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$$\begin{tikzcd} \mathrm{Ind}_{W_E}^{W_F}(\theta) \arrow[d, leftrightarrow] & \sigma_\theta \arrow[d, leftrightarrow, "\text{LLC for tori} + \chi\text{-datum}"] \\ \theta & \theta \end{tikzcd}$$

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Let π = a supercuspidal repn of $GL_n F$ with $\pi \cong \pi \otimes (\varepsilon \circ \det)$.

If for $\theta : E^\times \to \mathbb{C}^\times$ in gen pos and some constant $c \in \mathbb{C}^1$

$$\Theta_\pi(\gamma) = c \cdot \Theta_{\pi(\theta)}(\gamma) \qquad \text{for all } \gamma \in E^\times \text{ v.reg}$$

$\gamma \in \mathcal{O}_E^\times$, im in $\mathbb{F}_{q^n}^\times$
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then $\quad \pi \cong \pi(\theta) \quad$ (and $c = +1$).

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Rmk.

- If $n = \ell$, then Henniart gives a similar (simpler!) pf, even including E/F ram.
- But in general, such comparisons are much more subtle

(e.g. Harris–Taylor vs. Kaletha comparison: Oi–Tokimoto 2020)

Warning! I've lied a little bit...

<u>Counterexample</u>. $n=2$, $q=3$. res field of E is $\mathbb{F}_{q^2} = \mathbb{F}_9$.

- $\theta : \mathbb{F}_9^\times \to \mathbb{C}^\times$ $\qquad\qquad$ $\theta' : \mathbb{F}_9^\times \to \mathbb{C}^\times$

$\mathbb{F}_9^\times = \langle x \rangle$

$x \mapsto \zeta$ $\qquad\qquad$ $x \mapsto -\zeta$

$\uparrow$ prim 8th root of unity

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  $\{x, x^3\}$, $\{x^{-1}, x^{-3}\}$, $\{x^2, x^{-2}\}$

  and one can see: for $y = x, x^{-1}, x^2$,

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- This is the only failure, and as long as we assert $c=1$, then we're OK.

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for some $\mathbb{T}$ and $\theta : \mathbb{T}(\mathbb{F}_q) \to \mathbb{C}^\times$.

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!! $R_{\mathbb{T}}^{\mathbb{G}}(\theta)$ is irreducible $\iff$ θ is in general pos

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So $\left\{ \begin{array}{c} \theta : \mathbb{T}(\mathbb{F}_q) \to \mathbb{C}^\times \\ \text{in gen pos} \end{array} \right\} \Big/ W \hookrightarrow \left\{ \begin{array}{c} \text{irreps of} \\ \mathbb{G}(\mathbb{F}_q) \end{array} \right\}$.

Thm. (C-Oi) Assume $q \gg 0$. Let π = Irrep of $\mathbb{G}(\mathbb{F}_q)$.

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Henniart	C-Oi
$G = GL_n / F$	$\mathbb{G}$ = conn red / $\mathbb{F}_q$
$T = E^\times$ elliptic unram	$\mathbb{T}$ any max'l torus
π is s.c.	π is irred
all s.c. are "in gen pos"	not all irreps are "in gen pos"

OK for principal series too!

$\mathbb{T}$ spl

$q \gg 0 \quad \ldots \quad ?$

$q >> 0 \ldots$?

The precise thing we need is:

$$\frac{|\Pi(\mathbb{F}_q)|}{|\Pi(\mathbb{F}_q) \setminus \Pi(\mathbb{F}_q)^{rss}|} > 2 \cdot |W| \quad . \qquad (*)$$

$q >> 0 \ \ldots\ ?$

The precise thing we need is:

$$\frac{|T(\mathbb{F}_q)|}{|T(\mathbb{F}_q) \setminus T(\mathbb{F}_q)^{rss}|} > 2 \cdot |W| . \qquad (*)$$

- Everything in the bottom is an elt of a lower-rank torus, so LHS $\sim \dfrac{q^{\mathrm{rk}\, T}}{q^{< \mathrm{rk}\, T}}$. OTOH, RHS is indep of q.

So for $q >> 0$, $(*)$ holds.

$q >> 0$ ... ?

$$\mathbb{T}(\mathbb{F}_q)^{nrss} = \mathbb{T}(\mathbb{F}_q) \setminus (\mathbb{T}(\mathbb{F}_q) \cap \mathbb{G}(\mathbb{F}_q)^{rss})$$

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The precise thing we need is:

$$\frac{|\mathbb{T}(\mathbb{F}_q)|}{|\mathbb{T}(\mathbb{F}_q)^{nrss}|} > 2 \cdot |W| . \qquad (*)$$

- When $\mathbb{T}$ split, then W is big

$$(\text{e.g. } \mathbb{G} = GL_n, \ \mathbb{T}(\mathbb{F}_q) \cong (\mathbb{F}_q^\times)^n, \ |W| = n!)$$

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and if $\mathbb{T}$ is Coxeter, then W is even smaller

$(\text{e.g. } \mathbb{G} = Sp_4, \ \mathbb{T}(\mathbb{F}_q) \cong \ker(\mathbb{F}_{q^4}^\times \to \mathbb{F}_q^\times), \ |W| = 4)$

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  unless $\begin{cases} n=2, q=2 \\ \quad\;\; q=3 \\ n=3, q=2 \\ n=4, q=2 \end{cases}$

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Numericals for (some) exceptional gps

- $\mathbb{G} = G_2$, $\mathbb{T} =$ Coxeter

$|\mathbb{T}(\mathbb{F}_q)| = q^2 - q + 1$

$$|\mathbb{T}(\mathbb{F}_q)^{nrss}| = \begin{cases} 3 & \text{if } 3 \mid q+1 \\ 1 & \text{otherwise} \end{cases}$$

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$|W| = 12$

- $\mathbb{G} = E_6$, $\mathbb{T} =$ Coxeter

$|\mathbb{T}(\mathbb{F}_q)| = (q^4 - q^2 + 1)(q^2 + q + 1)$

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Numericals for (some) exceptional gps

$$\frac{|\mathbb{T}(\mathbb{F}_q)|}{|\mathbb{T}(\mathbb{F}_q)^{nrss}|} > 2|W| \text{ if:}$$

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Can we deal with small q? (i.e. does the conclusion of Thm hold even when $(*)$ is not satisfied?)

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- combination of tricks to handle small q for

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We haven't completed this in all cases yet, but I'll tell you about G_2 later.

Pf of Thm

Thm. (C-Oi) Assume $q \gg 0$. Let π = Irrep of $\mathbb{G}(\mathbb{F}_q)$.

If $\exists$ $\theta : \mathbb{T}(\mathbb{F}_q) \to \mathbb{C}^\times$ in gen pos and $c \in \mathbb{C}^1$ const. s.t.

$$\Theta_\pi(\gamma) = c \cdot \Theta_{R_{\mathbb{T}}^{\mathbb{G}}(\theta)}(\gamma) \quad \forall\, \gamma \in \mathbb{G}(\mathbb{F}_q)^{rss}$$

special case of DL char formula

$$= \begin{cases} c \cdot \sum_{w \in W} \theta^w(\gamma) & \text{if } Z_{\mathbb{G}}^\circ(\gamma) \underset{\mathbb{G}(\mathbb{F}_q)}{\sim} \mathbb{T} \\ 0 & \text{otherwise} \end{cases}$$

then $c \in \{\pm 1\}$ and $\pi \cong c \cdot R_{\mathbb{T}}^{\mathbb{G}}(\theta)$.

Pf of Thm Notation: we'll write $\langle \pi_1, \pi_2 \rangle_A = \frac{1}{|\mathbb{G}(\mathbb{F}_q)|} \sum_{g \in A} \Theta_{\pi_1}(g) \overline{\Theta_{\pi_2}(g)}$

for any subset $A \subseteq \mathbb{G}(\mathbb{F}_q)$.

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Pf of Thm

Notation: we'll write $\langle \pi_1, \pi_2 \rangle_A = \frac{1}{|\mathbb{G}(\mathbb{F}_q)|} \sum_{g \in A} \Theta_{\pi_1}(g) \overline{\Theta_{\pi_2}(g)}$

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Task: estimate to show that this sum is $\neq 0$.

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- Cauchy-Schwarz $\Rightarrow$ $|\quad| \leq \langle R^G_T(\theta), R^G_T(\theta) \rangle_{nrss}$.

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Task: estimate to show that this sum is $\neq 0$.

- Cauchy-Schwarz $\Rightarrow \left| \quad \right| \leq \langle R_{\mathbb{T}}^{\mathbb{G}}(\theta), R_{\mathbb{T}}^{\mathbb{G}}(\theta) \rangle_{nrss}$.
- Therefore to have sum $\neq 0$, it suffices to have

$$\langle R_{\mathbb{T}}^{\mathbb{G}}(\theta), R_{\mathbb{T}}^{\mathbb{G}}(\theta) \rangle_{nrss} < \frac{1}{2}. \qquad (**)$$

Pf of Thm (cont'd)

We now know that $\pi \cong c \cdot R^{\mathbb{G}}_{\mathbb{T}}(\theta)$ if

$$\langle R^{\mathbb{G}}_{\mathbb{T}}(\theta), R^{\mathbb{G}}_{\mathbb{T}}(\theta) \rangle_{nrss} < \frac{1}{2} \text{ holds.} \qquad (**)$$

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$\Big\|$ DL character formula
+ orthogonality of Green fns.

$$\frac{1}{|\mathbb{T}(\mathbb{F}_q)|} \sum_{\substack{w \in W \\ t \in \mathbb{T}(\mathbb{F}_q)^{nrss}}} \theta(t) \overline{\theta^w(t)} < \frac{1}{2} \qquad (**)$$

Pf of Thm (cont'd)

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Δ-ineq: $$\left| \qquad \right| \leq \frac{|T(\mathbb{F}_q)^{nrss}|}{|T(\mathbb{F}_q)|} \cdot |W|$$

Pf of Thm (cont'd)

We now know that $\pi \cong c \cdot R_{\mathbb{T}}^{\mathbb{G}}(\theta)$ if

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Δ-ineq: $\Big| \overbrace{\qquad\qquad}^{\downarrow} \Big| \leq \dfrac{|\mathbb{T}(\mathbb{F}_q)^{nrss}|}{|\mathbb{T}(\mathbb{F}_q)|} \cdot |W|$

So $(*) \Rightarrow (**) \Rightarrow \pi \cong c \cdot R_{\mathbb{T}}^{\mathbb{G}}(\pi)$.

$$\left(\frac{|\mathbb{T}(\mathbb{F}_q)|}{|\mathbb{T}(\mathbb{F}_q)^{nrss}|} > 2|W| \right)$$

(and $\therefore c = \{\pm 1\}$) $\square$

Small characteristics for $\mathbb{G} = G_2$, $\mathbb{T} =$ Coxeter

Recall: $|\mathbb{T}(\mathbb{F}_q)| = q^2 - q + 1$, $|W| = 6$.

(*) holds if $q \neq 2, 3, 5$.

Small characteristics for $G = G_2$, $T =$ Coxeter

Recall: $|T(\mathbb{F}_q)| = q^2 - q + 1$, $|W| = 6$.

(*) holds if $q \neq 2, 3, 5$.

$q = 5$. $T(\mathbb{F}_5) \cong \mu_{21}$ and the 5-Frob acts by $x \mapsto x^{-4}$.

There are 3 W-orbits of chars of μ_{21}.

Check "by hand" that (**) holds for each orbit.

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There are 3 W-orbits of chars of μ_{21}.

Check "by hand" that (**) holds for each orbit.

<u>$q = 2$.</u> $T(\mathbb{F}_2) \cong \mu_3$ and there are no char. in gen pos,

so the Thm is vacuous in this case.

__$q=3$.__ $\pi(\mathbb{F}_3) \cong \mu_7$. There is one W-orbit of θ in gen pos.

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$$= -1$$

and $\dim \pi = \dfrac{1}{\dim \mathrm{St}_{G_2(\mathbb{F}_3)}} \cdot \left| \mathbb{G}(\mathbb{F}_3)\text{-conj. cl of } \pi \right|$

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$\divideontimes$ So π is not unipotent and $\pi \cong \pm R_{\mathbb{T}}^{\mathbb{G}}(\theta)$.