

A strong Henniart identity
for reductive groups over finite fields.

Charlotte Chan (MIT)

joint with Masao Oi (Kyoto)

Supercuspidal representations of p-adic GL_n

F = nonarch local field, \mathfrak{W} = unit, \mathcal{O}_F = ring of integers, \mathbb{F}_q = res field, char p

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Q: Can we compare these parametrizations?

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s.t. $\pi \cong \pi \otimes (\varepsilon \otimes \det)$
where $\varepsilon: F^\times \rightarrow \mathbb{C}^\times$ any
char. s.t. $\ker(\varepsilon) = \text{Nm}_{E/F}(E^\times)$

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In this setting, $\theta \mapsto \pi(\theta)$ was also constr by Gerardin

(who further gave a constr for general G
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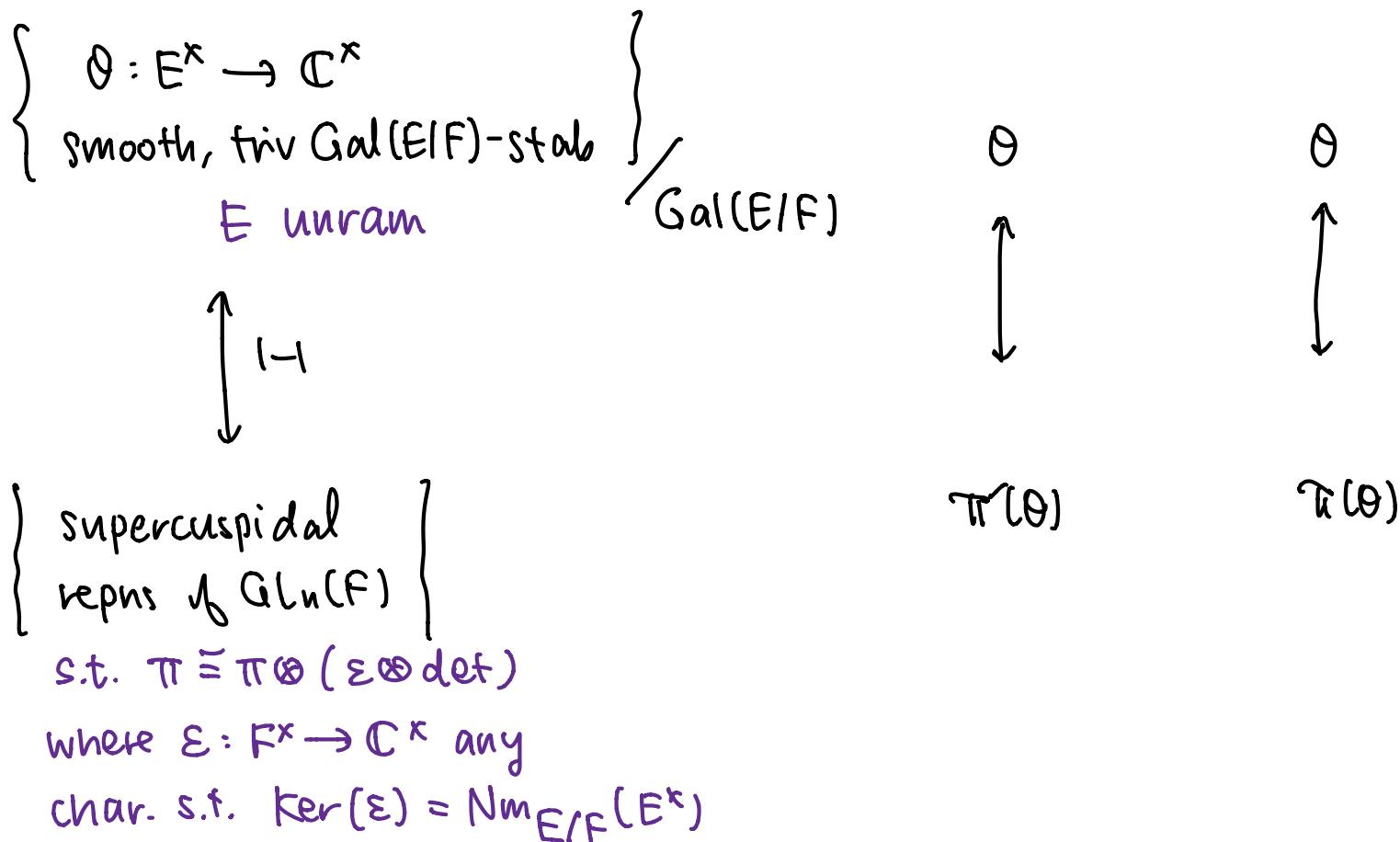
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Thm. (Henniart) Let $\theta: E^\times \rightarrow \mathbb{C}^\times$ smooth, triv $\text{Gal}(E/F)$ -stab.

Then $\pi'(\theta) \cong \pi(\theta w)$, where $w = \text{unram char of } E^\times$ s.t.
 $w(\infty) = (-1)^{n-1}$

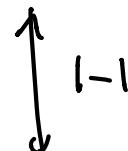
(Note: $w = \mathbb{I}$ if n odd, $w = \text{unram quad. char. if } n \text{ even}$)

Local Langlands for GL_n



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$\left\{ \begin{array}{l} n\text{-dim'l irrep } \sigma \\ \text{obj } W_F \text{ s.t. } \sigma \cong \sigma \otimes \varepsilon \end{array} \right\}$



$\left\{ \begin{array}{l} \theta: E^\times \rightarrow \mathbb{C}^\times \\ \text{smooth, friv Gal}(E/F)\text{-stab} \\ E \text{ unram} \end{array} \right\}$

$\text{Gal}(E/F)$

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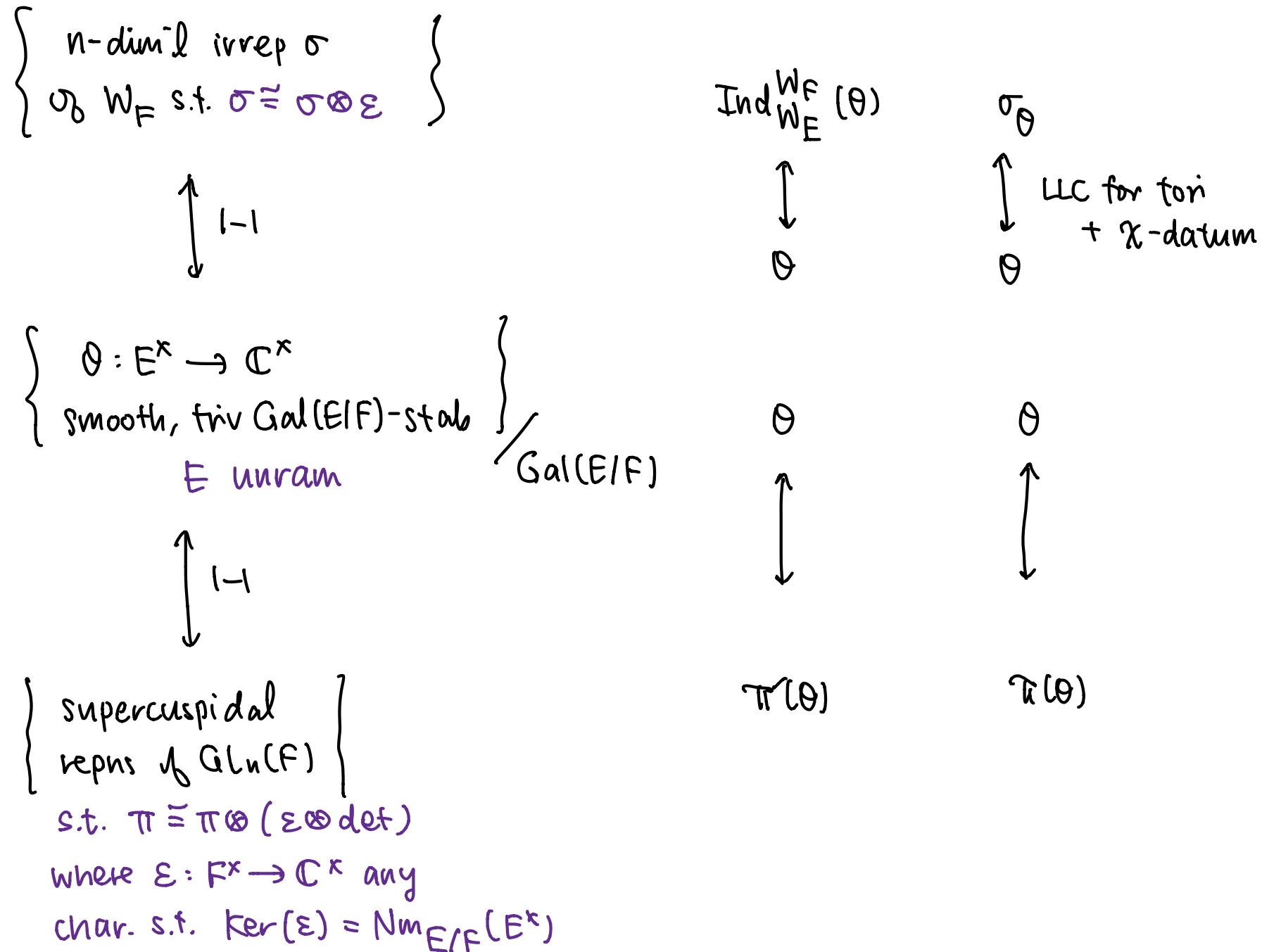
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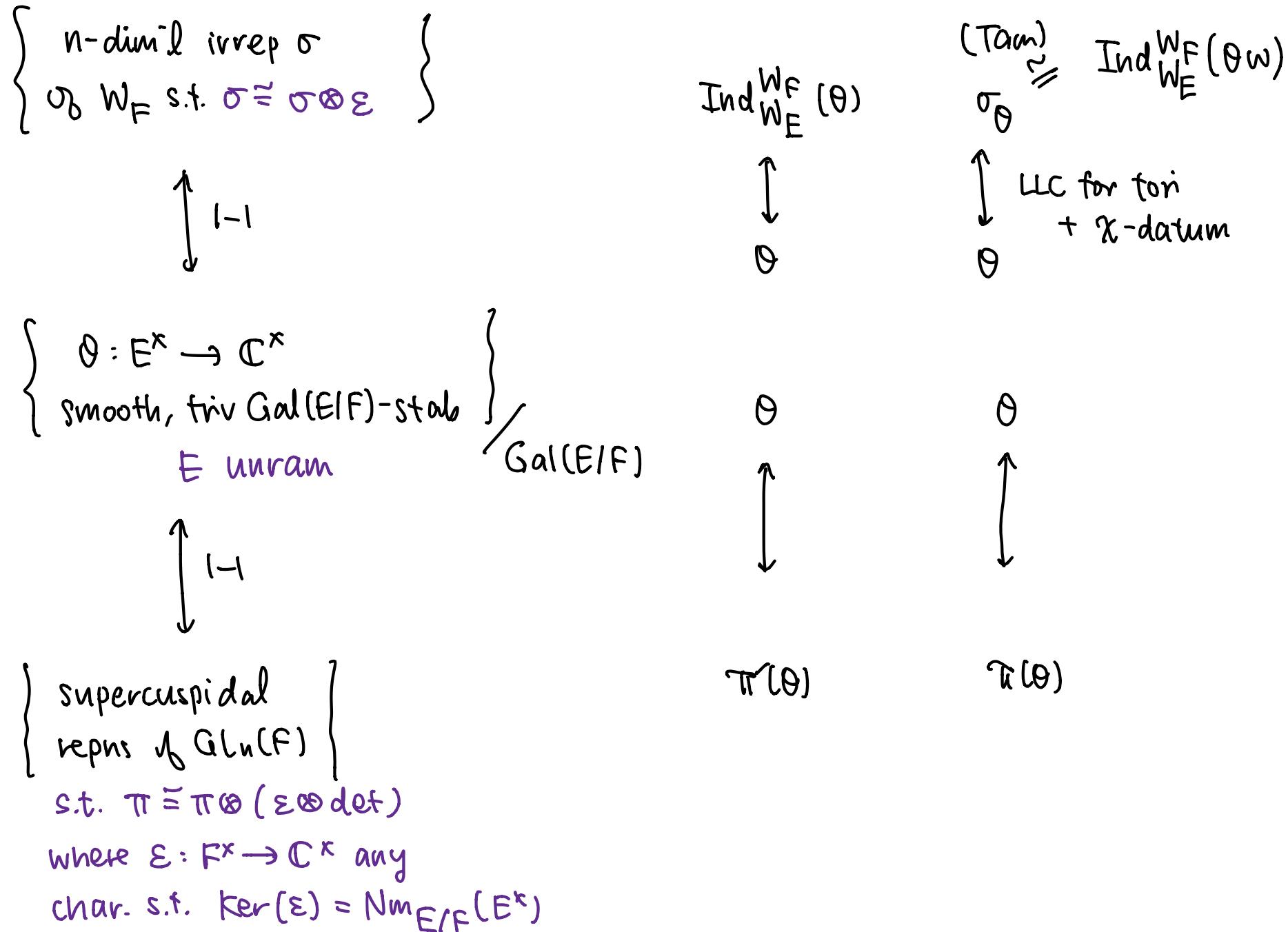
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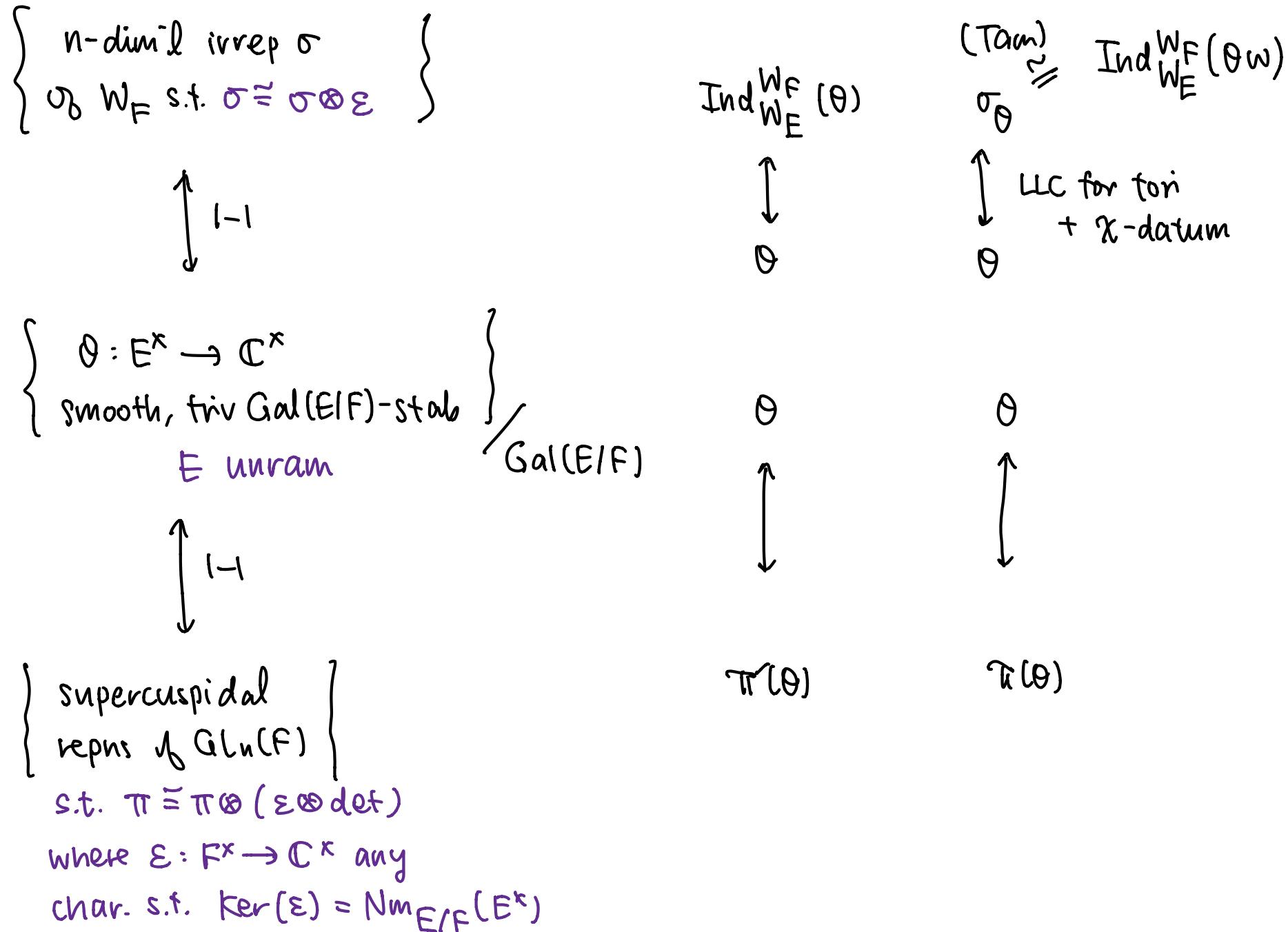
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Thm (Henniart)

Let π = a supercuspidal repn of $GL_n F$ with $\pi \cong \pi \otimes (\varepsilon \circ \det)$.

If for $\theta: E^\times \rightarrow \mathbb{C}^\times$ in gen pos and some constant $c \in \mathbb{C}^\times$

$$\textcircled{4}_{\pi}(\gamma) = c \cdot \textcircled{4}_{\pi(\theta)}(\gamma) \quad \text{for all } \gamma \in E^\times \text{ v. reg}$$

$\gamma \in O_E^\times$, im in $\mathbb{F}_{q^n}^\times$
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- Rmk.
- If $n=l$, then Henniart gives a similar (simpler!) pf, even including E/F ram.
 - But in general, such comparisons are much more subtle (e.g. Harris-Taylor vs. Kaletha comparison :)
 (Oi-Tokimoto 2020)

Warning! I've lied a little bit...

Counterexample. $n=2$, $q=3$. res field $\cup_b E$ is $\mathbb{F}_{q^2} = \mathbb{F}_q$.

$$\bullet \quad \theta: \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times \quad \theta': \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$$

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$\langle x \rangle$

$$x \mapsto \zeta$$

↑ prim 8th
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- The im $\sqrt[3]{E}$ v. reg elts in E^\times form 3 $\text{Gal}(\mathbb{F}_q / \mathbb{F}_3)$ -orbits:

$$\{x, x^3\}, \quad \{x^{-1}, x^{-3}\}, \quad \{x^2, x^{-2}\}$$

and one can see: for $y = x, x^{-1}, x^2$,

$$\sum \theta^w(y) = - \sum \theta'^w(y).$$

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- This is the only failure, and as long as we assert $c=1$, then we're OK.

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\mathbb{G} = conn red. grp / \mathbb{F}_q , $\Pi \hookrightarrow \mathbb{G}$ max'l \mathbb{F}_q -rat'l torus

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Deligne-Lusztig: All irreps of $\mathbb{G}(\mathbb{F}_q)$ appear in $R_{\mathbb{F}}^{\mathbb{G}}(\theta)$
for some Π and $\theta: \Pi(\mathbb{F}_q) \rightarrow \mathbb{C}^\times$.
 $R_{\mathbb{F}}^{\mathbb{G}}(\theta) := H_c^*(DLvar)[\theta]$

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So $\left\{ \begin{array}{l} \theta: \Pi(\mathbb{F}_q) \rightarrow \mathbb{C}^\times \\ \text{in gen pos} \end{array} \right\} \xrightarrow[W]{} \left\{ \begin{array}{l} \text{irreps of} \\ \mathbb{G}(\mathbb{F}_q) \end{array} \right\}$.

Thm. (C-0i) Assume $q \gg 0$. Let $\pi = \text{Irrep of } \mathbb{G}(\mathbb{F}_q)$.

If $\exists \theta: \pi(\mathbb{F}_q) \rightarrow \mathbb{C}^\times$ in gen pos and $c \in \mathbb{C}^\times$ const. s.t.

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then $c \in \{\pm 1\}$ and $\pi \cong c \cdot R_{\mathbb{F}}^{\mathbb{G}}(\theta)$.

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$q >> 0 \dots ?$

$q \gg 0 \dots ?$

The precise thing we need is:

$$\frac{|\pi(\mathbb{F}_q)|}{|\pi(\mathbb{F}_q) \setminus \pi(\mathbb{F}_q)^{\text{rss}}|} > 2 \cdot |w|. \quad (*)$$

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- Everything in the bottom is an elt of a lower-rank forms,
so LHS $\sim \frac{q^{\text{rk } \pi}}{q^{<\text{rk } \pi}}$. OTOH, RHS is indep of q .

So for $q \gg 0$, $(*)$ holds.

$q > 0 \dots ?$

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The precise thing we need is:

$$\frac{|\pi(\mathbb{F}_q)|}{|\pi(\mathbb{F}_q)^{\text{rss}}|} > 2 \cdot |W|. \quad (*)$$

- When π split, then W is big

(e.g. $G = GL_n$, $\pi(\mathbb{F}_q) \cong (\mathbb{F}_q^\times)^n$, $|W| = n!$)

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Numericals for (some) exceptional gps

- $G = G_2, \quad \mathbb{T} = \text{Coxeter}$

$$|\mathbb{T}(\mathbb{F}_q)| = q^2 - q + 1$$

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We haven't completed this in all cases yet, but I'll tell you about G_2 later.

Pf of Thm

Thm. (C-0i) Assume $q \gg 0$. Let $\pi = \text{Irrep of } G(\mathbb{F}_q)$.

If $\exists \theta: \pi(\mathbb{F}_q) \rightarrow \mathbb{C}^\times$ in gen pos and $c \in \mathbb{C}^\times$ const. s.t.

$$\bigoplus_{\gamma \in \pi} (\gamma) = c \cdot \bigoplus_{R \in \mathbb{F}_q[G]} R^G(\theta) \quad \forall \gamma \in G(\mathbb{F}_q)^{\text{reg}}$$

special case of
DL char formula

$$= \begin{cases} c \cdot \sum_{w \in W} \theta^w(\gamma) & \text{if } z_G^0(\gamma) \sim_{G(\mathbb{F}_q)} T \\ 0 & \text{otherwise} \end{cases}$$

then $c \in \{\pm 1\}$ and $\pi \cong c \cdot R^G_T(\theta)$.

Pf of Thm

Notation: we'll write $\langle \pi_1, \pi_2 \rangle_A = \frac{1}{|G(\mathbb{F}_q)|} \sum_{g \in A} \Theta_{\pi_1}(g) \overline{\Theta_{\pi_2}(g)}$
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- Therefore:

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Task: estimate \uparrow to show that this sum is $\neq 0$.

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Task: estimate \uparrow to show that this sum is $\neq 0$.

- Cauchy-Schwarz $\Rightarrow | \uparrow | \leq \langle R_{\pi}^G(\theta), R_{\pi}^G(\theta) \rangle_{nrss}$.
- Therefore to have sum $\neq 0$, it suffices to have

$$\langle R_{\pi}^G(\theta), R_{\pi}^G(\theta) \rangle_{nrss} < \frac{1}{2}. \quad (\#)$$

Pf of Thm (cont'd)

We now know that $\pi \cong c \cdot R_{\overline{F}}^G(\theta)$ if

$$\langle R_{\overline{F}}^G(\theta), R_{\overline{F}}^G(\theta) \rangle_{\text{nrss}} < \frac{1}{2} \quad \text{holds.} \quad (\star\star)$$

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|| DL character formula
+ orthogonality of Green fns.

$$\frac{1}{|\pi(\mathbb{F}_q)|} \sum_{\substack{w \in W \\ t \in \pi(\mathbb{F}_q)^{\text{nrss}}}} \theta(t) \overline{\theta^w(t)} < \frac{1}{2} \quad (\star\star)$$

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$$\Delta-\text{ineq: } | \underbrace{\quad \downarrow \quad}_{\substack{\text{---} \\ \text{---}}} | \leq \frac{|\pi(\mathbb{F}_q)^{\text{nrss}}|}{|\pi(\mathbb{F}_q)|} \cdot |W|$$

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$$\text{So } (*) \Rightarrow (**) \Rightarrow \pi \cong c \cdot R_{\overline{\mathbb{F}}}^G(\pi).$$

$$\left(\frac{|\Pi(\mathbb{F}_q)|}{|\Pi(\mathbb{F}_q)^{\text{nrss}}|} > 2|W| \right)$$

(and $\therefore c = 1 \pm i$)

□

Small characteristics for $G = G_2$, $\Pi = \text{Coxeter}$

Recall: $|\Pi(\mathbb{F}_q)| = q^2 - q + 1$, $|W| = 6$.

(*) holds if $q \neq 2, 3, 5$.

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$q=5$. $\Pi(\mathbb{F}_5) \cong \mu_{21}$ and the 5-Frob acts by $x \mapsto x^{-4}$.

There are 3 W -orbits of chars of μ_{21} .

Check "by hand" that (***) holds for each orbit.

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$q=2$. $\Pi(\mathbb{F}_2) \cong \mu_3$ and there are no char. in gen pos,
so the Thm is vacuous in this case.

$q=3$. $\pi(\mathbb{F}_3) \stackrel{\cong}{=} \mu_7$. There is one W -orbit of θ in gen pos.

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(using tricks sim to n pf
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DL prove: If π is a unip rep, then

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and $\dim \pi = \frac{1}{\dim \text{St}_{G_2(\mathbb{F}_3)}} \cdot |\text{ } G(\mathbb{F}_3)\text{-conj. cl of } \pi|$

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not div by 3

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DL prove: If π is a unip. rep, then

$$\begin{aligned} \langle \pi, R_T^G(\text{triv}) \rangle &= \bigoplus_{s \in \pi} s \quad \text{for any } s \in \pi(\mathbb{F}_q)^{\text{rss}} \\ &= -1 \end{aligned}$$

$$\begin{aligned} \text{and } \dim \pi &= \frac{1}{\dim \text{St}_{G_2(\mathbb{F}_3)}} \cdot |\text{ } G_2(\mathbb{F}_3)\text{-conj. cl of } \pi| \quad \text{not div by 3} \\ &= \frac{1}{q^{\# \text{pos roots}}} \cdot \frac{|G_2(\mathbb{F}_3)|}{|W| \cdot |\pi(\mathbb{F}_q)|} = \frac{\overbrace{q^6(q^6-1)(q^2-1)}^{\notin \mathbb{Z}}}{q^6(q^2-q+1) \cdot 6} \end{aligned}$$

※ So π is not unipotent and $\pi \cong \pm R_T^G(\theta)$.