Hamiltonian reduction for affine Grassmannian slices

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1. Affine Grassmanian slice

G semisimple $\quad Gr = G(\mathbb{C}[t,t^{-1}])/G(\mathbb{C}[t])$

We have spherical $G[t]$-orbits,

$$Gr^\lambda = G[t]t^\lambda \quad \lambda \text{ dominant cowt}$$

and transverse $G_1[t^{-1}]$-orbits:

$$W_\mu = G_1[t^{-1}]t^\mu \quad \mu \text{ dom.}$$

Then we form $\quad W_\mu^\lambda = \overline{Gr^\lambda} \cap W_\mu, \quad \mu \leq \lambda$

$G = SL_n \quad \lambda = n\omega_1, \quad \mu = 0, \quad W_\mu^\lambda = \mathcal{N}_{sl_n}$

Properties [Braverman-Finkelberg-Nakajima, KWWY]

① W_μ^λ is affine Poisson with conical symplectic singularities, $\dim W_\mu^\lambda = 2\rho(\lambda-\mu)$

② $T \curvearrowright W_\mu^\lambda$ with unique fixed pt t^μ and attracting locus $\overline{Gr^\lambda} \cap S^\mu \quad S^\mu = N[t,t^{-1}]t^\mu$

③ There is an integrable system $W_\mu^\lambda \xrightarrow{\Psi_\mu^\lambda} \mathbb{A}^{\rho(\lambda-\mu)}$

④ W_μ^λ is a Coulomb branch of a quiver gauge theory

⑤ The quantization of W^λ_μ is a truncated shifted Yangian Y^λ_μ.

⑥ $H_{top}(\overline{Gr^\lambda} \cap S^{-\mu}) \cong V(\lambda)_\mu$ [MV] $\qquad H_{top}((\Phi^\lambda_\mu)^{-1}(0)) \cong (Un_- \otimes V(\lambda))_\mu$

If μ is not dominant, then W^λ_μ requires a more complicated definition.

$W_\mu = U_1[t^{-1}]T_1[t^{-1}]t^\mu U_{-,1}[t^{-1}] \qquad W^\lambda_\mu = \overline{G[t]t^\lambda G[t]} \cap W_\mu$

inside $G[t,t^{-1}]$ [Bullimore - Dimofte - Gaiotto]

All the above properties hold for these generalized slices. (For ④, need $V(\lambda)_\mu \neq 0$)

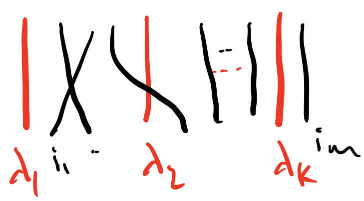
Y^λ_μ is a subquotient of the Yangian for μ dom.

The integrable system quantizes to a GT-subalgebra of Y^λ_μ

$G = SL_n$, $Y^{n\omega_1}_0 = U\mathfrak{sl}_n / Z_+$, usual GT-subalgebra

For λ, μ we have

T^λ_μ KLRW algebra



$\lambda = \lambda_1 + \dots + \lambda_k$

$\sum \alpha_{i_k} = \lambda - \mu$

<u>Theorem</u> [KTWWY] (simplified version)

Y^λ_μ-GT mod $\cong$ T^λ_μ-mod $\rightsquigarrow (Un_- \otimes V(\lambda))_\mu$

Y^λ_μ-cat $\mathcal{O}$ $\cong$ $(T^\lambda_\mu)_+$-mod $\rightsquigarrow V(\lambda)_\mu$

Can compute the number of simple GT-modules for $\mathfrak{sl}_n$ - an open problem!

Question

We have functors $E_i : T^\lambda_\mu\text{-mod} \to T^\lambda_{\mu+\alpha_i}\text{-mod}$.

How to describe the resulting $Y^\lambda_\mu\text{-GTmod} \to Y^\lambda_{\mu+\alpha_i}\text{-GTmod}$?

2. Hamiltonian reduction for slices

Define $\mathbb{G}_a \circlearrowright W^\lambda_\mu$ by $a \cdot g = [x_i(a) g]$, $\Phi_i : W^\lambda_\mu \to \mathbb{C}$ moment map

There are multiplication maps

$$W^0_{\mu_1} \times W^0_{\mu_2} \longrightarrow W^0_{\mu_1+\mu_2}$$

which quantize to

$$Y_{\mu_1+\mu_2} \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$$

defined in [FKPRW]

Theorem [KPW]

We have an iso $W^\lambda_\mu /\!/ \mathbb{G}_a = \Phi_i^{-1}(1)/\mathbb{G}_a \cong W^\lambda_{\mu+\alpha_i}$

① in fact $W^\lambda_{\mu+\alpha_i} \times W^0_{-\alpha_i} \xrightarrow{\sim} \Phi_i^{-1}(\mathbb{C}^\times) \subset W^\lambda_\mu$ given by mult.

② This quantizes to an iso:

$$Y^\lambda_{\mu+\alpha_i} \otimes Y^0_{-\alpha_i} \leftarrow Y^\lambda_\mu[\Phi_i^{-1}]$$

(conj outside of type A)

given by comultiplication

$W^0_{-\alpha_i} \cong T^*\mathbb{C}^\times$, $Y^0_{-\alpha_i} \cong D(\mathbb{C}^\times)$

Unfortunately this theorem does not lead to a functor between categories of GT-modules.

3. Coulomb branches

H reductive group, V rep.

Higgs $\quad T^*V /\!\!/ H$

Coulomb $\quad M_C(H,V) = \operatorname{Spec} H_\bullet\left(\operatorname{Maps}\left(D \cup_{D^\times} D, \left[V/H\right]\right)\right)$

Given G, λ, μ, get quiver gauge theory H, V

$G = SL_4 \quad \lambda = 2\omega_2 + \omega_3 \quad \mu = \lambda - \alpha_1 - 3\alpha_2 - 2\alpha_3$

$$\begin{tikzcd} & \boxed{2} \arrow[d] & \boxed{1} \arrow[d] \\ 1 \arrow[r] & 3 \arrow[r] & 2 \end{tikzcd}$$

$H = GL_1 \times GL_3 \times GL_2$

Theorem [BFN]

$W^\lambda_\mu \cong M_C(H,V) \qquad Y^\lambda_\mu \cong A(H,V)$

Every $M_C(H,V)$ has a quantization $A(H,V)$

Given any H, V and any $\xi : \mathbb{C}^\times \to H$, we can relate $A(H,V)$ and $A(L_\xi, V^\xi)$

Theorem [KWWY]

There is a functor

$$A(H,V)\text{-}\underset{\text{mod}}{GT} \to A(L_\xi, V^\xi)\text{-}\underset{\text{mod}}{GT}$$

If we take H, V so that $M_C(H,V) = W^\lambda_\mu$

and $\xi : \mathbb{C}^\times \to H = \prod GL_{v_i}$

is $\mathbb{C}^\times \to GL_{v_i}$ $\quad s \mapsto \begin{bmatrix} s & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$

Then $L_\xi = \prod_{j \neq i} GL_{v_j} \times GL_{v_i - 1} \times \mathbb{C}^\times$

and $M_C(L_\xi, V^\xi) = W^\lambda_{\mu+\alpha_i} \times W^0_{-\alpha_i}$

$\rightsquigarrow$ Y^λ_μ - GT mod $\longrightarrow$ $Y^\lambda_{\mu+\alpha_i} \otimes Y^0_{-\alpha_i}$ - GT mod

Theorem

The diagram

$$\begin{tikzcd}
Y^\lambda_\mu\text{-GT mod} \arrow[r, phantom, "\cong"] \arrow[d] & T^\lambda_\mu\text{-mod} \arrow[d, "E_i"] \\
Y^\lambda_{\mu+\alpha_i}\text{-GT mod} \arrow[r, phantom, "\cong"] & T^\lambda_{\mu+\alpha_i}\text{-mod}
\end{tikzcd}$$

defined using KLR algebras

commutes