Compactifying the category of D-modules on Bun_G

Plan

① Compactifications of D-modules
② Is Bun_G proper?
③ Is T^*Bun_G proper?
④ (Main result) Putting things together

(in progress, partly joint with Gaitsgory)
Main question

\[ \text{Drinfeld-Gaitsgory:} \quad \text{D-Mod}(\text{Bun}_G) \text{ is "nice" similar to} \quad \mathcal{O}_X \text{-Mod} \quad X=g \text{-projective} \]

\[ \text{compactify} \quad \mathcal{O}_X \text{-mod} \quad \overline{X} = \text{projective} \]

\[ C = \text{smooth proj. curve/C} \]
\[ G = \text{reductive group} \]
\[ \text{Bun} = \text{Bun}_G(C) = \text{stack of G-bundles on C} \]
Compactifying D-modules (good filtrations) \(X=\text{smooth}\)

Reminder \(M=\text{coherent}\)

\(D_x\)-module

Good filtration is

\[ M = \bigoplus_{d} M^{\leq d} \quad \mathcal{D} = \bigoplus_{d} \mathcal{D}^{\leq d} \]

Reformulation (Rees)

\[ \oplus M^{\leq d} \quad \oplus \mathcal{D}^{\leq d} \]

coherent, graded module.

\(\text{gr}(M)\) is \(\text{gr}(D_x)\)-coherent

\(\text{Spec } \text{gr}(D_x)\)

\(\text{gr}(M)\) is \(T^*X\) coherent.
Geometry

Rees $D_x = D_x^h = C[t]$ - algebra of diff.
operators with parameter $t$

Summary: Good filtrations:
$D_x$-mod $\rightarrow (D_{x,t^k}$-mod)$^\infty$

$D_x^h = \langle \mathcal{O}_x, T_x \rangle$

$[\tau, f] = t^h D_x f$ $\forall f \in \mathcal{O}$

$\deg f = 0$, $\deg t = \deg \tau = 1$
Geometry

Rees $D_x = D_{\mathfrak{t}} = C[t]$-algebra of diff. operators with parameter $t$

Summary: Good filtrations:
\[
\text{D}_x\text{-mod} \longrightarrow (\text{D}_{x+\mathfrak{t}}\text{-mod})_{\mathbb{G}_m}
\]

\[D_\mathfrak{t} = \langle 0_x, J_x \rangle\]

\[\exists f \in D_\mathfrak{t} \text{ such that } f \in \mathcal{O}_x\]

\[\deg f = 0, \quad \deg J = \deg \mathfrak{t} = 1\]

\[
\left(\left(\mathbb{T}^* \times \mathcal{O}\right)/\mathbb{G}_m\right) / \mathbb{G}_m.
\]

Analogy:
\[
\mathbb{T}^* X \longrightarrow \left(\mathbb{T}^* \times \mathcal{A}\right)/\mathbb{G}_m
\]

Better
\[
\left(\left(\mathbb{T}^* \times \mathcal{A}\right)/\mathbb{T}^* X\right)/\mathbb{G}_m
\]

fiberwise projectivization

(g if $X$ is projective, actual compactification)!
Summary - Exercise

\(X = \text{smooth projective}\)

Put

\[ C := \frac{(D_\mathbb{A} \text{-Mod})^{\mathbb{G}_m}}{\text{irrelevant}} \]

\(C\) is proper and \(D_x\text{-Mod}\) is its localization.

\(C\) is proper:

\[ \dim \text{Ext}^i(F,G) < \infty \]

for \(F,G \in C\) compact.
Goal: Make it work for $X = \text{Bun}_G$

not projective unless $G$ is torus.

Hope (Simpson) (g classical limit)

\[ (T^*\text{Bun}_G) = \{G\text{-bundles with}\} \]
\[ \{ \text{connection on } C \} \]
\[ \cap \]
\[ \{ G\text{-bundles with } \} \]
\[ \{ \text{th-connections} \} \]

stability conditions.

\[ E \text{ is proper and } \text{Dr}_X\text{-Mod is} \]

for $F,G \in E$ compact

Put

$C := (\text{Dr}_X\text{-Mod})^{\text{Gm}} / \text{irrelevant}$

$C$ is proper and $\text{Dr}_X\text{-Mod}$ is

its localization

$\dim \text{Ext}^1(F,G) < \infty$
\textbf{Is }\text{Bun}_G\text{ "proper"?}

(Drinfeld - Gaitsgory)

Harker - Narasimhan - Shatz

stratification:

\[\text{Bun}^\beta_G \to \text{Bun}^\alpha_G \to \cdots\]

1. \text{Bun}^\lambda_G \text{ is q. compact}

2. For \(d \gg 0\),

\[j^! : \text{D-Mod} (\text{Bun}^\delta_G) \to \text{D-Mod} (\text{Bun}^\beta_G)\]

preserves coherence - \text{unwound}

\[\text{holonomic!}\]

\textbf{Toy example: }j : A - [10]/A/_{6m} \to A/_{6m}

\[\lim D\text{-Mod}(\text{Bun}_G)\]

\textbf{Corollary: }D\text{-Mod}(\text{Bun}_G) \text{ is compactly generated by}

\[\{j^! F | F \text{ coherent } D\text{-mod}\} \text{ on } \text{Bun}_G\text{ for } d \gg 0\]
2. Is $Bun$ "proper"?

(Drinfeld-Gaitsgory)

Harder: Narasimhan-Seshadri stratification:

$Bun_G^{\beta} \to Bun_G \to \ldots$

1. $Bun_G^{\beta} \to q.$ compact

2. For $\ell \gg 0,$

$j_*: D\text{-}Mod(Bun_{\beta}) \to D\text{-}Mod(Bun_{\beta})$

preserves coherence

Problem Fails for $D_{\mathcal{H}}\text{-}Mod$.

Even $\ell = 0$: $O_{T^* Bun_{\mathcal{G}}} - \text{Mod}$ is not compactly generated.
3. Is $T^\ast \mathbb{B}_n$ proper?

GIT: Kirwan-Ness stratification, Teleman, Halpern-Leistner

$Y^{\leq \gamma} \subset Y^{\leq \beta} \subset Y_{\text{compact}}$

$Y^{\leq \gamma}$ is (homologically) proper

$\dim \text{Ext}^i(F_1, F_2) < \infty$
Main result

A coherent extension functor

\[ \text{D}^+_\text{c}(\text{Bun}^{<L}) \to \text{D}^+_\text{c}(\text{Bun}^{<B}) \]

(similar to Halpern-Leistner's)

Remark: Works without \( \mathfrak{g}_m \)-action
Corollary "correct" category

\[ \lim_{\ell \to \infty} \text{D}_{\text{ti}} \text{-Mod}(\text{Bun}_G) = D_{\text{ti}} \text{-Mod}(\text{Bun}_{G,\ell}) \]

Properties

1. \( C \) is compactly generated
2. \( C_{\text{Gm}} \) is proper
3. \( D_{\text{-Mod}}(\text{Bun}_G) \to C_{\text{Gm}} \)
4. Interesting even if \( t_i = 0 \) and more: ??
5. Independent of choices
6. Hecke - invariant
7. Compatible with \( j \) Hodge \( D \)-modules
8) Compatible with Hitchin: $D^+_h \text{-Mod} (\text{Bun}_c) \rightarrow D^+_h \text{-Mod} (\text{Bun}_G)$

9) $(D^+_h \text{-Mod} (\text{Bun}_c) / E_{\text{Milp}}) \\ \text{already proper.}$

Remark: Ignored safety
Key example:

\[ \mathbf{A}^3 \to \mathbf{A}^1 \to \{0\} \]

Weakly En-equivariant

\[ \mathbf{D}_n \text{-Mod.} \]

\( \delta \) has many possible weak structures: \( \delta^{-1}, \delta^0, \delta^1 \)

\[ \mathbf{D}_n = k \langle x \rangle, \quad [3, x] = t_1 \]

\[ \delta = \mathbf{D}_n / \mathbf{D}_n \cdot x \]

\[ \mathbf{M} = \mathbf{D}_n / \mathbf{D}_n (3x) \]

Exercise:

\[ \text{Ext}^i (\delta \langle i \rangle \mathbf{M}) = 0 \quad i > 0 \]

\[ \text{Ext}^i (\mathbf{M}, \delta \langle i \rangle) = 0 \quad i < 0 \]
**Toy application (Gaitsgory)**

$F_1, F_2 = \text{coherent } D_{\text{Bun}_G} \text{-mod.}$

$\text{Sing Supp}(F_2) \subset \text{Nilp}.$

Then $\dim \text{Ext}^i(F_1, F_2) < \infty$

**Proof:**

$j^* F_2$ is compact

$F_1 = j^* \tilde{F}_1$ for compact $\tilde{F}_1$ (good field)

adjustment $j^*, j_*, \text{properness}$