Compactifying the category of D-modules on Bun_G

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(in progress, partly joint with Gaitsgory)

Plan

1. Compactifications of D-modules
2. Is Bun_G proper?
3. Is $T^*\mathrm{Bun}_G$ proper?
4. (Main result) Putting things together

Main question

C = smooth proj. curve$/\mathbb{C}$

G = reductive group

$\mathrm{Bun} = \mathrm{Bun}_G(C)$ = stack of G-bundles on C

Drinfeld-Gaitsgory:

compactly generated

D-Mod(Bun_G) is "nice"

localization

compactly generated

proper

?

similar to

$\mathcal{O}_X$-Mod

X = q. projective

compactify

$\mathcal{O}_{\bar{X}}$-mod

$\bar{X}$ = projective

① Compactifying D-modules (good filtrations)

X = smooth

Reminder M = coherent as a D-module

D_X-module

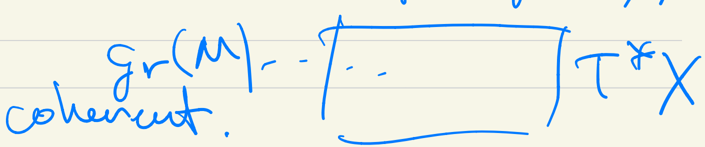
Good filtration is

$M = \bigcup_d M^{\leq d} \circlearrowleft D_X = \bigcup_d D_X^{\leq d}$ order $\leq d$

s.t. $gr(M)$ is $gr(D_X)$-coherent

{ geometrically

Spec $gr(D_X)$



Reformulation (Rees)

$\text{Rees}\, M = \bigoplus M^{\leq d}$ $\quad$ $\text{Rees}\, D_X = \bigoplus D_X^{\leq d}$.

coherent, graded module.

1. Geometry

$\mathrm{Rees}\, D_X = D_\hbar =$

$\mathbb{C}[\hbar]$- algebra of diff. operators with parameter $\hbar$

Summary: Good filtrations:

$$D_X\text{-mod} \dashrightarrow \left(D_{X,\hbar}\text{-mod}\right)^{\mathbb{G}_m}$$

$$\boxed{\begin{gathered} D_\hbar = \langle \mathcal{O}_X, T_X \rangle \\ [\tau, f] = \hbar\, \partial_\tau(f) \quad f \in \mathcal{O},\ \tau \in T \\ \deg f = 0,\ \deg \hbar = \deg \tau = 1 \end{gathered}}$$

① Geometry

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$\mathbb{C}[\hbar]$- algebra of diff. operators with parameter $\hbar$

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$D_\hbar = \langle \mathcal{O}_X, T_X \rangle$

$[\tau, f] = \hbar D_\tau(f)$ $\quad f \in \mathcal{O}$, $\tau \in T$

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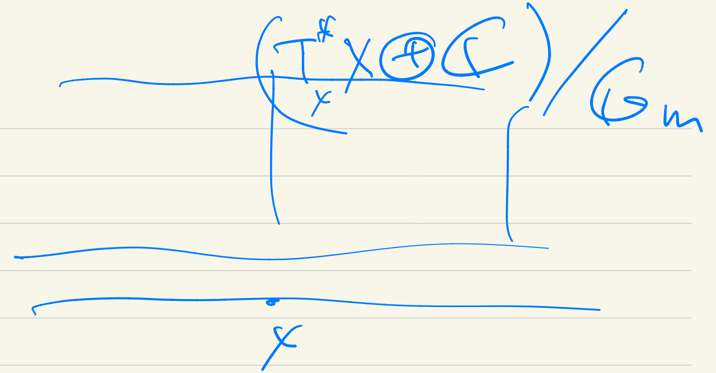

Analogy:

$$T^*X \dashrightarrow (T^*X \times \mathbb{A}^1)/\mathbb{G}_m$$

Better

$$((T^*X \times \mathbb{A}^1) - X)/\mathbb{G}_m$$

fiberwise 'projectivization' of T^*X.

(if X is projective – actual compactification!)

① Summary - Exercise

X = smooth projective

$t_\hbar, T_X$ act nilpotently

Put

$\mathcal{C} := (D_\hbar\text{-Mod})^{\mathbb{G}_m} / \text{irrelevant}$

$\mathcal{C}$ is proper and D_X-Mod is its localization

$\boxed{\begin{array}{l}\mathcal{C} \text{ is proper:} \\ \dim \operatorname{Ext}^i(\mathcal{F}, G) < \infty \\ \text{for } \mathcal{F}, G \in \mathcal{C} \text{ compact}\end{array}}$

① Summary - Exercise

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Put

$$\mathcal{C} := (D_\hbar\text{-Mod})^{\mathbb{G}_m} / \text{irrelevant}$$

$\mathcal{C}$ is proper and D_X-Mod is its localization

$\mathcal{C}$ is proper:
$\dim \mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) < \infty$
for $\mathcal{F}, \mathcal{G} \in \mathcal{C}$ compact

Goal Make it work for $X = \mathrm{Bun}_G$

not projective unless G is torus.

Hope (Simpson) (q. classical limit)

$$\begin{pmatrix} \text{Twisted} \\ T^*\mathrm{Bun}_G \end{pmatrix} = \left\{ \begin{matrix} G\text{-bundles with} \\ \text{connection on } C \end{matrix} \right\}$$

$$\bigcap$$

$$\left\{ \begin{matrix} G\text{-bundles with} \\ \hbar\text{-connections} \end{matrix} \right\}$$

+ stability conditions.

② Is Bun_G "proper"?

not even q. compact!

(Drinfeld - Gaitsgory)

Harder-Narasimhan-Shatz stratification:

$$\mathrm{Bun}_G^{\leq\alpha} \xrightarrow{j} \mathrm{Bun}_G^{\leq\beta} \subset \cdots$$

1. $\mathrm{Bun}_G^{\leq\alpha}$ is q. compact

2. For $\alpha \gg 0$,

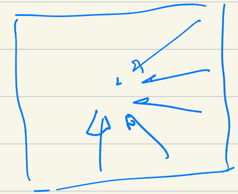
$j_*: \text{D-Mod}(\mathrm{Bun}^{\leq\alpha}) \to \text{D-Mod}(\mathrm{Bun}^{\leq\beta})$

D-modules

preserves coherence - even without holonomicity!

Model

~~Toy~~ example: $j: \mathbb{A}^n - \{0\}/\mathbb{G}_m \hookrightarrow \mathbb{A}^n/\mathbb{G}_m$



$\underset{\overrightarrow{j_*}}{\lim}$ " $\text{D-Mod}(\mathrm{Bun}_G^{\leq\alpha})$ ".

Corollary: $\text{D-Mod}(\mathrm{Bun}_G)$ is compactly generated.

by $\left\{ j_* \mathcal{F} \;\middle|\; \begin{array}{l} \mathcal{F} = \text{coherent D-mod} \\ \text{on } \mathrm{Bun}_G^{\alpha} \quad \alpha \gg 0 \end{array} \right\}$.

② Is Bun_G "proper"?

(Drinfeld–Gaitsgory)

Harder-Narasimhan-Shatz stratification:

$$\mathrm{Bun}_G^{\leq \alpha} \xhookrightarrow{j} \mathrm{Bun}_G^{\leq \beta} \subset \cdots$$

1. $\mathrm{Bun}_G^{\leq \alpha}$ is q. compact

2. For $\alpha \gg 0$,

$j_* : D\text{-Mod}(\mathrm{Bun}^{\leq \alpha}) \to D\text{-Mod}(\mathrm{Bun}^{\leq \beta})$

preserves coherence

Problem **Fails** for $D_\hbar$-Mod.

Even $\hbar = 0$: $\mathcal{O}_{T^*\mathrm{Bun}_G}$-Mod is

not compactly generated.

$Y_{||}$ / $\mathbb{G}_m$

③ Is "$T^*\mathrm{Bun}_G$ proper"?

GIT: Kirwan-Ness stratification, Teleman, Halpern-Leistner

$Y^{\le\alpha} \subset Y^{\le\beta} \subset \cdots$
j

— $Y^{\le\alpha}$ are q. compact

$Y^{\le\alpha}/\mathbb{G}_m$ is (cohomologically) proper

$\forall$ coherent F_1, F_2 on $Y^{\le\alpha}/\mathbb{G}_m$

$\dim \mathrm{Ext}^i(F_1, F_2) < \infty$

— $Z = Y^{\le\beta} - Y^{\le\alpha}$

choices

$\mathrm{Coh}(Z)$
$\|$ semiorthogonal
$\langle \mathrm{Coh}(Z)^L, \mathrm{Coh}(Z)^R \rangle$

Z

$\mathrm{Coh}(Y^{\le\beta})$
$\|$ semi-orthogonal.
$\langle \mathrm{Coh}(Y^{\le\beta})^L, \mathrm{Coh}(Y^{\le\beta})^\alpha, \mathrm{Coh}(Y^{\le\beta})^R \rangle$

Supp on Z — $0 \quad \mathrm{Coh}(Y^{\le\alpha}). \quad 0$ — Supp on Z

j_* (under each of the three components)

④ Main result

$\exists$ coherent extension functor

$$D_\hbar\text{-Mod}(\mathrm{Bun}^{\le\alpha}) \longrightarrow D_\hbar\text{-Mod}(\mathrm{Bun}^{\le\beta})$$

(similar to Halpern-Leistner's).

~~quantization~~

$\mathrm{Bun}^{\le\alpha} \subset \mathrm{Bun}^{\le\beta} \subset \dots$

$$\mathcal{C}_\beta$$
$$\text{"}D_\hbar\text{-Mod}(\mathrm{Bun}^{\le\beta})\text{"}$$
$$\| \quad \text{semi orthog.}$$
$$\langle \mathcal{C}^L_\beta, \ \mathcal{C}^\alpha_\beta, \ \mathcal{C}^R_\beta \rangle$$

$$\begin{array}{c} \mathcal{C}^\alpha_\beta \\ \downarrow \wr \\ \mathcal{C}_\alpha \end{array}$$

extension functor on D-modules

Remark Works without $\mathbb{G}_m$-action

Corollary "Correct" category

$$D_\hbar\text{-Mod}(\mathrm{Bun}_G) = \lim_{\substack{\longrightarrow \\ \lambda \gg 0}} D_\hbar\text{-Mod}(\mathrm{Bun}_G^{\leq \lambda})$$

$\mathcal{C} :=$ this extension.

Properties

① $\mathcal{C}$ is compactly generated

② $\mathcal{C}^{\mathbb{G}_m}$ is proper

③ $D\text{-Mod}(\mathrm{Bun}_G) \hookrightarrow \mathcal{C}^{\mathbb{G}_m}$

④ Interesting even if $\hbar = 0$

and more: ??

⑤ Independent of choices

⑥ Hecke-invariant

⑦ Compatible with ~~j~~ Hodge D-modules.

⑧ Compatible with Hitchin : $D_{\hbar}\text{-Mod}(\mathrm{Bun}_G) \overset{?}{\to} \left(D_{\hbar}\text{-Mod}(\mathrm{Bun}_G)\right)_{\mathcal{N}ilp}$

0 fiber of Hitchin

⑨ $\left(D_{\hbar}\text{-Mod}(\mathrm{Bun}_G) \overset{\mathcal{C}}{\underset{\shortparallel}{}} \Big/ \mathcal{C}_{\mathcal{N}ilp} \right)^{\mathbb{G}_m}$

already proper.

Remark Ignored safety

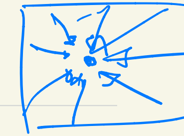
Key example:

$$\mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1 \to \{0\}.$$

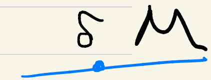
Weakly $\mathbb{G}_m$-equivariant $D_\hbar$-Mod.

δ has many possible weak structures: $\ldots, \delta\langle -1\rangle, \delta\langle 0\rangle, \ldots$

$D_\hbar = k\langle x, \xi\rangle$, $[\xi, x] = \hbar$



$$\delta = D_\hbar / D_\hbar \cdot x$$



$$M = D_\hbar / D_\hbar(\xi x).$$

Exercise:

$\mathrm{Ext}^\bullet(\delta\langle i\rangle, M) = 0 \quad i \geq 0$

$\mathrm{Ext}^\bullet(M, \delta\langle i\rangle) = 0 \quad i < 0$

Toy application (Gaitsgory)

$\mathcal{F}_1, \mathcal{F}_2$ = coherent D_{Bun_G}-Mod.

$\mathrm{Sing\,Supp}(\mathcal{F}_2) \subset \mathcal{N}ilp.$

Then $\dim \mathrm{Ext}^i(\mathcal{F}_1, \mathcal{F}_2) < \infty$

$\mathcal{F}_2 \quad \mathcal{F}_2$

$$D\text{-Mod}(Bun_G) \underset{j^*}{\overset{j_*}{\rightleftarrows}} \text{Compactification}$$

$$\|$$

$$\left(D_\hbar\text{-Mod}(Bun) \big/ D_\hbar\text{-Mod}(Bun)_{Nilp} \right)^{Com}$$

Proof:

$j_* \mathcal{F}_2$ is compact

$\mathcal{F}_1 = j^* \tilde{\mathcal{F}}_1$ for compact $\tilde{\mathcal{F}}_1$ (good filtration)

adjunction j^*, j_*, properness