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Representing Hochschild cohomology of Soergel bimodules

$G/\mathbb{F}_q$ – reductive, adjoint type (knot theory: $G = PGL_n$).
k – field, char $k = 0$, $k = \bar{k}$ $\quad (k = \bar{\mathbb{Q}}_\ell)$

$\mathbb{S}$Bim $\quad \mathfrak{g}^\vee \supset \mathfrak{h}^\vee/k$
W – Weyl group, S – simple refl. $\quad \left(S_n \curvearrowright \mathfrak{h}^\vee, \ S = \{(12), (23), \dots, (n-1\,n)\} \right)$
$R = k[\mathfrak{h}^\vee] = \mathrm{Sym}[\mathfrak{h}]$

$\deg \mathfrak{h} = 2$.

R-mod-R – category of graded R-bimodules
Monoidal category, $\otimes_R$ – monoidal str.
$\mathbb{S}\mathrm{Bim}(W, \mathfrak{h}^\vee) \subset R\text{-mod-}R$ full additive subcat., closed under summands, $\otimes_R$, grading shifts, containing $B_s := R \otimes_{R^s} R(1)$. $s \in S$
Isom. classes of indec. $\rightsquigarrow B_w, w \in W$

Rouquier complexes: $Br = Br(W, S)$
braid group on n-strands. $(G = PGL_n)$
$b \in Br \rightsquigarrow F_b \in Ho(\mathbb{S}\mathrm{Bim}(W, \mathfrak{h}^\vee))$
$b_s, s \in S \longmapsto \Delta_s = \underline{B_s} \xrightarrow{m} R(1), \ \nabla_s = R(-1) \to \underline{B_s}$
$b_s \longmapsto R$

Hochschild cohomology:
$M \in R\text{-mod-}R$
$HH^i(M) = \mathrm{Ext}^i_{R\text{-mod-}R}(R, M)$.
Goal: find $X_k \in Ho(\mathbb{S}\mathrm{Bim}(W, \mathfrak{h}^\vee))$
for $M \in \mathbb{S}\mathrm{Bim}$, $HH^k(M) = \mathrm{Hom}_{Ho(\mathbb{S}\mathrm{Bim})}(X_k, M)$

Example 1: $X_0 = R$
Example 2: $X_{top} = X_{n-1} \doteq F_{w_0}^{\otimes 2} = $ Full Twist
$G = PGL_n$; up to grading shift; lift of the longest elt

[Gorsky – Hogancamp – Mellit – Nakagane in type A]

Motivation: Khovanov – Rozansky homology.
L – link in S^3, $L = \bar{\beta}$, $\beta \in Br_n$. (type A)
$HHH^k(L) = HH^k(F_\beta) \underset{def}{=}$
$= H^\bullet(\dots \to HH^k(F_\beta^n) \to HH^k(F_\beta^{n+1}) \to \dots)$
[GHMN] result above categorifies a result of Kalman, relating coef's of HOMFLY-PT polynomials of $\bar{\beta}$ & $\overline{w_0^2 \beta}$.

First answer:
$HH^\bullet(M)$ can be computed using the Koszul resolution $K^\bullet$ of $R \in R\text{-mod-}R$
$K^\bullet$ is a complex of free $R \otimes_k R$-modules ← not in $\mathbb{S}$Bim
$HH^\bullet(M) = H^\bullet(\underline{\mathrm{Hom}(K^\bullet, M)})$
complex of Hom's
Fact: $M \in \mathbb{S}\mathrm{Bim}$, $R \otimes R$-action factors through $R \otimes_{R^W} R$-action.
$K^\bullet_S := K^\bullet \otimes_{R \otimes R} R \otimes_{R^W} R$ $\quad \in C^\bullet(\mathbb{S}\mathrm{Bim})$?
Each term of $K^\bullet_S$ is of the form
$B_{w_0}^{\oplus k_i}(r_i) \quad (B_{w_0} \simeq R \otimes_{R^W} R(\ell(w_0)))$

Theorem 1:
there is a t-structure on $Ho(\mathbb{S}\mathrm{Bim})$ w. the cohomology functor ${}^p\mathcal{H}^\bullet_{w_0}$, such that $HH^k(M) \doteq \mathrm{Hom}({}^p\mathcal{H}^{-k}_{w_0}(K^\bullet_S), M)$.
We have, in particular, ${}^p\mathcal{H}^0_{w_0}(K^\bullet_S) \doteq R$.
($\doteq$ – up to grading shift) $\quad {}^p\mathcal{H}^{top}_{w_0}(K^\bullet_S) \doteq F_{w_0}^{\otimes 2}$.
X_k – repr. $HH^k \simeq {}^p\mathcal{H}^k_{w_0}(K^\bullet_S)$

Geometry:
G – reductive / $\mathbb{F}_q$, $B \subset G$ – Borel (split)
$U \subset B$ – unipotent radical $\quad T$ – maximal torus.
$Ho(\mathbb{S}\mathrm{Bim})$ – "algebraic Hecke category"
Webster – Williamson geom. int. of HHH
$D^b_{mix}(B \backslash G/B) \simeq D^b_{mix, \widehat{mon}}(U^\vee \backslash G^\vee / U^\vee)$
[Bezrukavnikov – Yun] $\quad \hat{D}^b_m(B \backslash G / B)$
(unip.) monodromic = locally-constant along fibers of $G/U \xrightarrow{\pi} G/B$
completed = allow free-monodromic unipotent local systems
Example: $e = eB \in G/B$, 0-dim. B-orbit.
$T \simeq B/U \simeq \pi^{-1}(e)$.

$R \in \mathbb{S}\mathrm{Bim}$
$\delta_e \in D^b_{mix}(B \backslash G/B) \qquad \hat{\mathcal{L}} \in D^b_{mix, \widehat{mon}}(U \backslash G/U)$
$\hat{\mathcal{L}}$ – pro-local system on T corresp. to "infinite Jordan block".

B_w, Δ_w, ∇_w
free-monodromic tilting
$IC_w, j_{w!}\bar{\mathbb{Q}}_\ell, j_{w*}\bar{\mathbb{Q}}_\ell \qquad \tilde{\mathcal{T}}_w, \tilde{j}_{w!}\hat{\mathcal{L}}, \tilde{j}_{w*}\hat{\mathcal{L}}$
$\tilde{\Delta}_w \qquad \tilde{\nabla}_w$

$$\begin{tikzcd} T \times O_w \simeq \tilde{O}_w \arrow[r, hook, "\tilde{j}_w"] \arrow[d] & G/U \arrow[d] \\ O_w \arrow[r, hook, "j_w"] & G/B \end{tikzcd}$$
Bruhat cell

Both categories are monoidal via group convolution $*$.
The monodromic category $\hat{D}^b_m(B \backslash G / B)$ carries a perverse t-structure $({}^pD^{\le 0}, {}^pD^{\ge 0})$ w. cohomology functors ${}^p\mathcal{H}^\bullet$.
Define $({}^pD^{\le 0}_{w_0}, {}^pD^{\ge 0}_{w_0}) = (\tilde{\Delta}_{w_0} * {}^pD^{\le 0}, \tilde{\Delta}_{w_0} * {}^pD^{\ge 0})$
with cohomology functor ${}^p\mathcal{H}^\bullet_{w_0}$.
${}^p\mathcal{H}^\bullet_{w_0}(X) = \tilde{\Delta}_{w_0} * {}^p\mathcal{H}^\bullet(\tilde{\nabla}_{w_0} * X)$.
$\tilde{\mathcal{T}}_{w_0}$ – "big projective".
$U \backslash G/U \xrightarrow{p} \dfrac{U \backslash G/U}{T}$ ← adjoint action

Theorem 2
1) $K^\bullet_S \rightsquigarrow p^* p_! \tilde{\mathcal{T}}_{w_0}$.
2) $\tilde{\mathcal{T}}_w$ are injective w.r.t. shifted t-str.
3) $H^\bullet(\underline{\mathrm{Hom}}(K^\bullet_S, \tilde{\mathcal{T}}_w)) \simeq \mathrm{Hom}({}^p\mathcal{H}^{-k}_{w_0}(p^* p_! \tilde{\mathcal{T}}_{w_0}), \tilde{\mathcal{T}}_w)$
$X_k \rightsquigarrow {}^p\mathcal{H}^{-k}_{w_0}(p^* p_! \tilde{\mathcal{T}}_{w_0})$.

Second answer (computation of X_k).
$D^b(U \backslash G/U)$
$\downarrow p$
$D^b(G/G) \xrightarrow{Av_{U!}} D^b\left(\dfrac{U \backslash G/U}{T}\right)$
adjoint action; (Harish-Chandra functor)
$Av_{U!}^{-1}(D^b_{mon})$ = derived category of character sheaves.
DCS.
$D^b(G/G)$ is a monoidal category w.r.t. to the !-group convolution $*$
$Av_{U!}$ is a monoidal functor.
$\hat{\delta}_G \in \mathrm{pro}DCS$, $p^* Av_{U!}(\hat{\delta}_G) = \tilde{\Delta}_e = \tilde{\nabla}_e$
(pro)unit in DCS
Key fact: $Av_{U!}$, restricted to DCS, is t-exact
perverse t-str. $\longmapsto$ w_0-shifted perverse t-str.
Bezrukavnikov – Finkelberg – Ostrik in char 0; Chen – Yom Din ℓ-adic

$\mathcal{N}^u \hookrightarrow G$ – unipotent variety.
$j: \mathcal{N}_{reg} \hookrightarrow \mathcal{N}^u$ – regular orbit.

Theorem 3 not a CS, supp. on $\mathcal{N}^u$
$j_* \mathbb{Q}_\ell = Av_{G!}\,\mathcal{L}_{\psi,U}$ – Whittaker sheaf
$Av_{U!}(\underline{j_* \bar{\mathbb{Q}}_\ell * \hat{\delta}_G}) \doteq p_! \tilde{\mathcal{T}}_{w_0}$ $\quad \in \mathrm{pro}DCS$ $\quad K^\bullet_S$
+ some shifts & Tate twists
Corollary 1:
${}^p\mathcal{H}^\bullet_{w_0}(p^* p_! \tilde{\mathcal{T}}_{w_0}) \simeq p^* Av_{U!}({}^p\mathcal{H}^\bullet(j_* \bar{\mathbb{Q}}_\ell) * \hat{\delta}_G)$
Theorem 4 (just type A on arXiv)
${}^p\mathcal{H}^k(j_* \bar{\mathbb{Q}}_\ell) \simeq Spr_{\Lambda^{n-1-k}\mathfrak{h}}$ – summand of the Springer sheaf corresp. to the exterior powers of $\mathfrak{h}$

Corollary 2:
$Av_{U!}(\underline{\bar{\mathbb{Q}}_{\mathcal{N}^u}} * \hat{\delta}_G) = \Delta_{w_0}^{*2}$
1) ${}^p\mathcal{H}^0_{w_0}(K^\bullet_S) \doteq R$
2) ${}^p\mathcal{H}^{top}_{w_0}(K^\bullet_S) \doteq F_{w_0}^{\otimes 2}$
3) ${}^p\mathcal{H}^k_{w_0}(K^\bullet_S)$ has a filtration by braid objects, categorifying the elem. symmetric polynomials in Jucys-Murphy braids (in type A).

Type A: $G = GL_n$ $\quad \tilde{\mathcal{H}}^{aff}_n \to \mathcal{H}_n \quad / k(v, v^{-1})$
extended affine Hecke algebra; finite Hecke algebra
$\tilde{\mathcal{H}}^{aff}_n = \langle t_s, s \in S, \ \theta_x, x \in \mathbb{X}$ – weight lattice $\rangle$
finite simple reflections $\quad \mathbb{X} = \langle e_1, \dots, e_n \rangle$
$\theta_i := \theta_{e_i}$
$\Phi: \tilde{\mathcal{H}}^{aff}_n \to \mathcal{H}_n$, $\Phi(t_s) = t_s$, $\Phi(\theta_1) = 1$
$\Phi(\theta_k) = J_k$ – Jucys-Murphy elt's in $\mathcal{H}_n$.
$J_2 = t_{s_1}^2$, $J_3 = t_{s_2} t_{s_1}^2 t_{s_2} \dots$
e_r – elementary symm. polyn.
$e_r(\theta_1, \dots, \theta_n) \in Z(\tilde{\mathcal{H}}^{aff}_n)$
$\Phi(e_r(\theta_1, \dots, \theta_n)) = e_r(J_1, \dots, J_n) \in Z(\mathcal{H}_n)$.
Objects X_k categorify $e_k(J_1, \dots, J_n)$ $\quad (X_k \oplus X_{k+1})$
Coherent geom. categor. of $\tilde{\mathcal{H}}^{aff}_n$
$D^b_{G,\mathcal{N}} Coh(St)$
$St = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ Steinberg variety
$\Delta: \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \mathfrak{g}$
$e_r(\theta_1, \dots, \theta_n) \rightsquigarrow \Delta_* \mathcal{O}(\Lambda^r V)$ – filtered by $\Delta_* \mathcal{O}(\mu)$
$GL_n = GL(V)$
μ runs through weights of $\Lambda^r V$
Conj.: $\tilde{\Phi}: D^b_{G,\mathcal{N}} Coh(St) \to D^b_{m, \widehat{mon}}(U \backslash G/U)$.
Gorsky – Rasmussen – Negut; Oblomkov – Rozansky; Coherent descr. of HHH
$\underline{\Delta_* \mathcal{O}(\Lambda^r V)} \longmapsto X_k$
$\underline{\mathcal{O}(\mu)} \longmapsto$ products of $\underline{J_i}$.
$\mathcal{O}_{St} \qquad X_k \qquad \tilde{\mathcal{T}}_{w_0}$
$Av_{U!}(\mathcal{H}^{-k} j_* \mathbb{Q}_\ell * \hat{\delta}) = X_k$.
$\underline{Q}$. is there a nice object in $\tilde{\Phi}^{-1}(K^\bullet_S)$.
$Av_{U!}(j_* \mathbb{Q}_\ell * \hat{\delta}) = K^\bullet_S$

$\underline{I}$. $j_* \mathbb{Q}_\ell = Av_{G!}\,\mathcal{L}_{\psi,U}$ $\qquad$ (work on $\mathfrak{g}$).
$\underline{II}$ $\quad FT\,\mathcal{L}_{\psi,U} = s^* \underline{\mathbb{Q}}_{\mathfrak{b}}$
regular in $\mathfrak{b}^-$
$Av_{G!}\, s^* \underline{\mathbb{Q}}_{\mathfrak{b}} = Av_{G!}\, \mathbb{Q}_S$ – Kostant slice.
easy to compute the stalk/ $t \in \mathfrak{g}^{rs}$
$\simeq H^\bullet(Z_G(t))$